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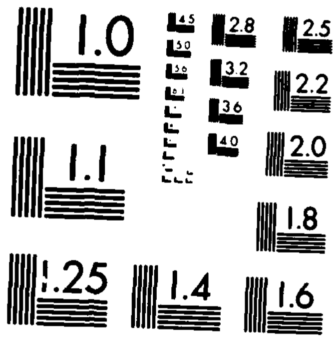

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A REVIEW OF IMPERFECTION SENSITIVITY OF STIFFENED SHELLS

Anamet Laboratories, Inc.
Applied Mechanics Division
San Carlos, California

February 1984

Interim Report for Period December 1982 - November 1983

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Flight Dynamics Laboratory
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A short history is related about how imperfections (i.e., deviations from the assumed theoretical form of a shell structure) came to be regarded as the major contributor to the experimental scatter observed in shell buckling tests, and the discrepancy between experimentally determined buckling loads and those determined by classical linear theory. J.M.T. Thompson's insight into the relationship between imperfection sensitivity and optimality of shell structures with respect to load configurations is brought out. The significant contributions to understanding elastic postbuckling behavior of thin stiffened shells and how this work relates to imperfection sensitivity are reviewed. The implications of this work and its applicability to current engineering design and analysis are assessed. The effects of geometric imperfections as well as those induced by boundary conditions or multiple load cases on the postbuckling behavior of shells are discussed. Added discussions by Profs. Singer and van Neut are also presented.

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BLOCK 18 (continued)

Elastic Stability
Structural Stability
Unstiffened Shells
Catastrophe Theory
Postbuckling
Shell Buckling
Buckling Loads

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1.0 INTRODUCTION

The stiffened shell is perhaps the most utilized structure in industry today. Since the early 1930's, the aerospace and aircraft industry has used this class of shell to its fullest extent. However, as the shell structure is made thinner and lighter for a given loading, it becomes prone to a catastrophic collapse known as shell instability. The magnitude of the load that causes the shell to become unstable is significantly affected by the size and geometry of imperfections in the initial shape of the shell. Consequently, in order to ensure an efficient design, considerable research has been conducted in the United States and overseas to more fully understand the complex behavior of these structures.

The development of an understanding of the behavior of shell structures had its start around the turn of the century. Surveys by Flügge[109*], Langhaar[189], Nowinski[224], Naghdi[216] and Nash[220,221] clearly demonstrate that even after more than three quarters of a century of study, there is a continued interest in shell analysis and, in particular, elastic shell stability analysis. In fact, the interest has been so great that the total number of papers written on the subject exceeds six thousand, of which nearly one thousand were written in the last decade.

The interest in shell analysis has not only been associated with the aerospace industry, but also in the marine and energy fields, just to mention a few. For example, the Subcommittee on Shells for the Pressure Vessel Research Committee of the Welding Research Council has supported several studies for the analysis of shell structures. Also, a chronicle of support of the Pressure Vessel and Piping Division of the ASME has been collected in several ASME documents[20,21]. The author has also presented a limited survey of the imperfection sensitivity and postbuckling

*Superscript numbers denote references found at end of report.

of shells as part of the Pressure Vessel and Piping Decade of Progress of 1982[79]. A more in-depth review of buckling of shells as a general category has been given by Bushnell[67,68].

After all that has been written on the subject, one might assume that the subject matter has been amply covered. Some researchers have taken this attitude, but many design/analyst engineers find that they must still rely upon ingenuity and engineering instincts when evaluating the stability characteristics of shell structures which are subject to a specific complex loading condition. Perhaps the present state-of-the-art is too mathematical or abstract for the practicing engineer. Perhaps the methods of analysis are incomplete with respect to the understanding of the instability of shells. This is more eloquently presented by Koiter[174] in his discussion of Truesdell and Noll's remarks[306] on the purpose of the elastic stability literature. Koiter states:

"Our first aim, to show the real purpose of existing theories and current research in elastic stability, is by far our easiest task. Structural stability is in fact one of the most important criteria in the design of many engineering structures. Our knowledge in this field is admittedly far from complete, in particular for structures in which inelastic behaviour or dynamic loading conditions are essential features, but the theory of elastic stability, however incomplete it may be, constitutes an indispensable tool in achieving properly designed and efficient structures, capable of withstanding their service loads without catastrophic failures. It is quite true, of course, that many investigations of elastic stability lean heavily on more or less crude approximations and on heuristic physical arguments, not by choice, however, but only due to the absence of a rigorous theory which is at the same time capable of a reasonably simple application to the problem at hand. Elastic stability shares the fate of all engineering science that it cannot afford to await the development of fully rigorous mathematical theories before it deigns to consider the solution of stability problems arising in engineering practice. Its servant's role in engineering implies that it must try and solve these problems, if possible by rigorous analysis, if necessary by hook or by crook. A rough estimate today may be worth more than an accurate approximation a month from now."

At present, the dilemma that the practicing engineer faces is that he has only two choices in examining the stability of imperfect shells -- he must use either (1) complex computer programs which are based upon certain theories and therefore have certain limitations -- some specified, others not; or (2) simplified equations modified by semi-empirical (graphical) data developed from experiments that may or may not have been adequately controlled. Without a complete understanding of the subject matter, the engineer could easily misuse these computer programs and/or data, and misleading conclusions could be drawn. The dilemma is further aggravated because modern technology demands that more efficient structures be designed, sometimes beyond the imposed limitations of presently accepted data or computational techniques.

It is therefore the purpose of this review to illustrate the basis of the accepted theories of the imperfection sensitivity of shells and also to suggest works that may be helpful to the practicing engineer in gaining additional insight. Since the previously mentioned surveys give a rather broad description and chronicle of shell instability studies, it would appear to be appropriate for this review to present material which permits the practicing engineer to focus on the complexities involved in the so-called elementary stability theories and understand their limitations. In this way he may bridge the gap between his needs and those accepted theories. Because of the enormous volume of papers on just one subject, it may require the engineer to review several papers to gain the required understanding of a single concept. A summary is provided to aid in this evaluation. The review must necessarily present abbreviated forms of some rather cumbersome mathematical expressions. Their appearance is by no means intended to be complete, but rather a reminder of their complexity.

2.0 FUNDAMENTAL PRINCIPLES AND THEIR HISTORICAL DEVELOPMENT

2.1 Basic Concepts

The foundation of the analysis of thin shells of revolution was developed more than a century ago and Novozhilov[223] presented a rather comprehensive description of this development. The basic equations of equilibrium were developed by Aron[29] in 1874 and Love[195] in 1888. In these early developments it was shown that the main feature that distinguishes thin shell theory from the three-dimensional theory of elasticity is that the shell can be regarded as a two-dimensional surface. All fundamental variables are therefore dependent on two orthogonal curvilinear coordinates, ζ_1 and ζ_2, which define a point on a middle or reference surface. The description of this reference surface obeys the relationships of differential geometry. The governing equations can be greatly simplified if the reference surface is restricted to a surface of revolution. The key parameters that describe the shell of revolution's geometry are its two principal radii of curvature, R_1 and R_2. For practical shell applications, the coordinate system ζ_1, ζ_2 is usually oriented along the principal curvature directions. For a shell of revolution, the in-plane coordinates are in the meridional, where $d\zeta_1 = R_1 d\phi$, and the circumferential, $d\xi_2 = R_2 d\theta$, directions. The third orthogonal coordinate is ζ_3 direction and is in the direction of the outward normal.

The development of the strain-displacement relations for a general shell in terms of the orthogonal curvilinear coordinates is given by Love. These equations, as well as the remaining fundamental equations of thin shells, are presented by Kraus[182]. In his presentation, the accepted definitions are used to illustrate Reissner's version of the Love theory. Four representive shell theories are discussed which preserve Love's original assumptions.

The measure of deformation of the reference surface can be expressed through a set of strains and curvatures which are given in terms of the displacement vector of the middle surface and the geometric parameters of the surface. The reference surface strain and curvature create stresses throughout the shell. Integrating the stresses and their first moments about the reference surface give the in-plane or membrane stress resultants and the bending stress resultants. Consistent with the theory of Love's first approximation are the following four assumptions:

(1) The thickness of the shell, t, is small such that t/R_1 and $t/R_2 << 1$.

(2) The strains and displacements are small so that second order terms and higher may be neglected.

(3) The extensional stresses normal to the middle surface may be considered small with respect to the other stress components.

(4) Normals to the reference surface prior to deformation remain normal after deformation and have no extension.

The displacement of a point on the reference surface of the shell is expressed by u, v and w in the ζ_1, ζ_2 and ζ_3 directions, respectively; and the displacement of a point of a distance, Z, away from the reference surface can be given by:

$$\begin{aligned} \hat{u}_\phi &= u + Z\,\beta_\phi \\ \hat{u}_\theta &= v + Z\,\beta_\phi \\ \hat{w} &= w \end{aligned} \tag{1}$$

where β_ϕ and β_θ are the reference rotation about the surface coordinates ζ_2 and ζ_1, respectively. Some theories permit local rotation (twist, Φ) about the normal coordinate as well.

In order to fully understand the implication of Love's first approximation, the equations of equilibrium and the strain-displacement equations must be considered in conjunction with the relationship between the stresses and strains, such as Hooke's Law for a linear elastic material. Love's first assumption defines what is meant by thin. Although a precise definition is not available, usually a radius to thickness ratio $R/t > 10$ has been accepted by many. The second assumption assures linearity of the governing differential equations. The third and fourth assumptions require the normal stress and strain and the transverse shear stresses to be negligible. Usually, for homogeneous, orthotropic materials, there are three mutually perpendicular planes of elastic symmetry, and these planes also usually lie in the planes of principal curvature. Thus, the theories are applicable to orthotropic shells of revolution.

Hildebrand, Reissner and Thomas[131] expand on this theme and give a more complete description of the higher order approximate theories. Several other methods have been presented by a number of investigators in an attempt to improve the analysis of shells. An excellent discussion of the various derivations of the governing equations, their similarities and their differences, is given by Koga and Endo[168], and by Leissa[192]. Over fourteen different theories are examined by Leissa. These include Novozhilov, Love, Timoshenko[304], Byrne[70], Flügge[108], Goldenveizer[120], Lur'ye[198], Reissner[240], Vlasov[313], Sanders[254], Donnell[96], and Mushtari[214]. In Leissa's comparison, he shows that the usual result when adopting any of these theories is that inconsistent approximations are required in order to make any significant improvements over Love's first approximation.

With the fundamental linear shell equations addressed, attempts were made between 1900 and 1914 by Lorenz[194], von Mises[317], Southwell[281] and Timoshenko[303] to examine the critical buckling load of a thin circular cylindrical shell under axial compression. The linear equations governing the buckling of a cylinder in a membrane prestressed state of compression due to an axial load, P, can be given in the form:

$$E_1(u,v,w) = 0$$

$$E_2(u,v,w) = 0 \qquad (2)$$

$$D\nabla^4 w + \frac{C}{R^2}(v_{,\theta} + w + \nu R u_{,x}) = -\frac{P}{2\pi R} w_{,xx}$$

where E_1 and E_2 are equilibrium equations in the ζ_1 and ζ_2 directions in terms of u, v, and w; R is the radius of the cylinder, x is the meridional coordinate (synonymous with the ϕ direction), θ is the circumferential independent variable, and commas denote partial differentiation. The variables C and D are given by:

$$C = \frac{Et}{1-\nu^2}$$

$$D = \frac{Et^3}{12(1-\nu^2)}$$

where E is the modulus of elasticity, ν is Poisson's ratio. Also the biharmonic operator is defined as:

$$\nabla^4 w = w_{,xxxx} + \frac{2}{R^2} w_{,xx\theta\theta} + \frac{1}{R^4} w_{,\theta\theta\theta\theta} \qquad (3)$$

These three governing equations described in Eq. (2) constitute a linear eigenvalue problem, and the eigenvalue is related to P. As P is increased from zero, the cylinder will undergo an end shortening linearly related to P and accompanied by a uniform radial expansion due to the Poisson effects. When the minimum eigenvalue is reached, the cylinder displacement pattern will be at a bifurcation point. For larger values of P, the shell could continue the same deformation pattern as before, or it could exhibit a non-uniform radial buckling displacement that could be quite large. The linear eigenvalue formulation does not allow any study of the post-buckling behavior. The only information it provides is for minimum bifurcation buckling load and the associated buckling mode shape.

When experimental results became available it was noted that, although good agreement could be attained for structures comprised of columns and plates, poor correlations between the analytical and experimental buckling loads were obtained for the thin-walled circular cylindrical shell[245]. In attempts to reconcile the discrepancy between experiment and theory for cylindrical shells, Donnell[97] and Flügge[107,108] suggested certain boundary condition restraints at the onset of buckling, but they could not account for the differences in the predicted and observed buckled shapes. Donnell's treatment in 1934[97] suggested the employment of a large deflection theory, which included an initial geometric imperfection in the shape of the cylinder. The imperfection, denoted by $\overline{w}$, changes Eq. (2) into:

$$D\nabla^4 w + \frac{C}{R^2}\left(v_{,\theta} + w + \nu R u_{,x}\right) = -\frac{P}{2\pi R}\left(w_{,xx} + \overline{w}_{,xx}\right) \tag{4}$$

This new formulation is not an eigenvalue problem, and hence the cylinder will begin to deform in a general way, with bending, as soon as P is increased from zero. The imperfection creates a pseudo load. Von Karman and Tsien[314,316] demonstrated that curved shells can develop buckled deflections that are different from the original buckling mode with an imperfection amplitude that is several times the wall thickness. Von Karman used the mode shape data taken from the Donnell experiment to prove his point and suggested that states of equilibrium could exist at loads lower than classical.

During the early periods of research for cylindrical shells under axial load, a single path to shell collapse (limit or bifurcation point) was accepted. Postbuckling studies indicated a dramatic decrease in the load-carrying ability occurs at the bifurcation point and the cylinder undergoes a dramatic collapse. Figure 1 illustrates Donnell and Wan's results[98] for an imperfect cylinder under axial compression. For many cases where cylinders of differing thinness ratios (R/t) were tested, the normalized load-deflection curve of the shell (in terms of the classical load λ^* and strain ε_{c1}) essentially follows a linear curve until some limit point (p) is reached, and then the shell snaps into a

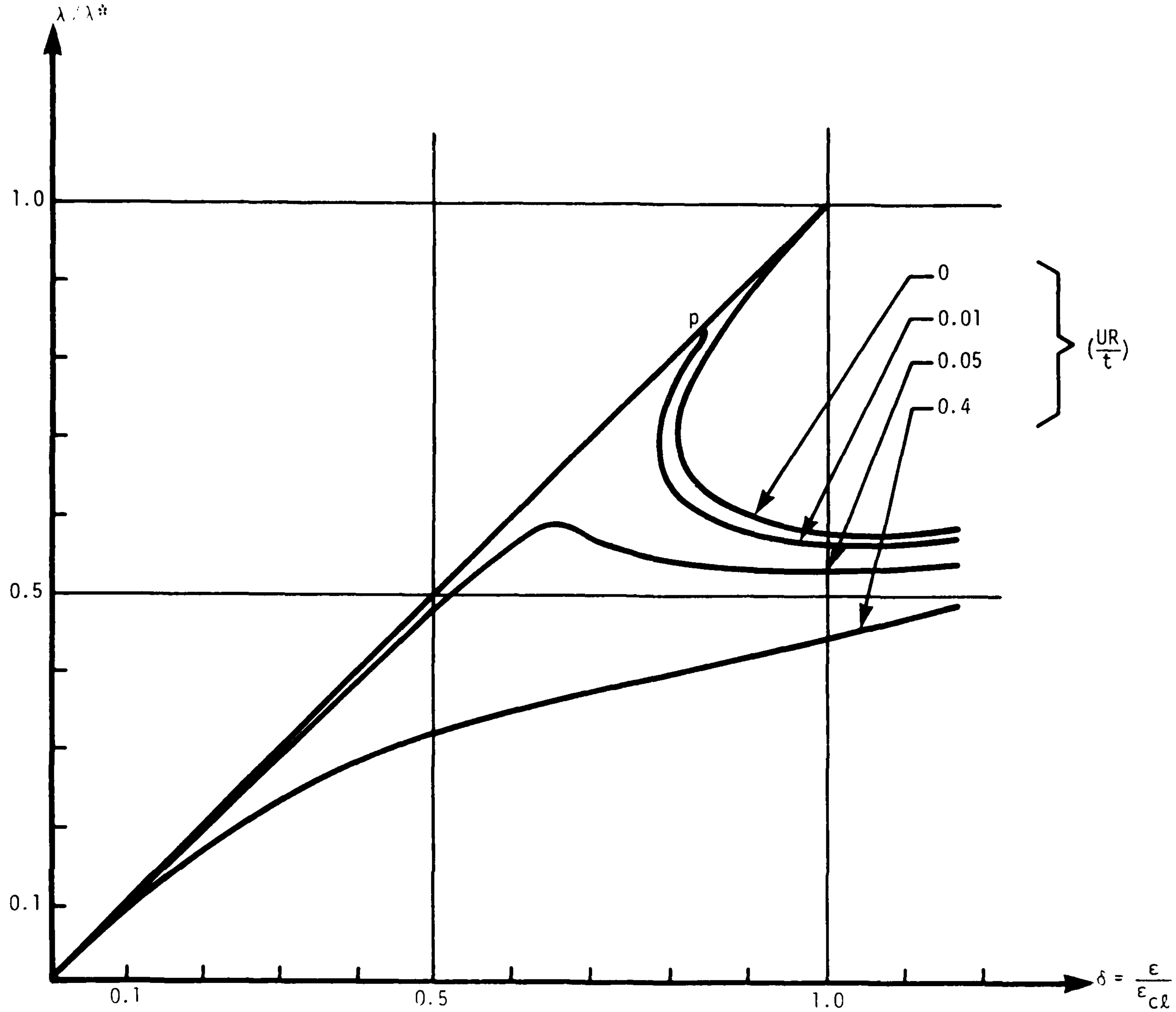


Figure 1. Load deflection curve-effects of imperfections.

postbuckled state. As the magnitude of the imperfection is increased (in terms of an uneveness factor, U), a marked reduction in the maximum load-carrying capacity of the shell is experienced. The imperfection can become so large that no snap-like behavior is experienced and the shell simply has an increasing nonlinear behavior with load, as indicated by the curve for UR/t = 0.4. The variation of the load-carrying capacity is much less sensitive to imperfections in the postbuckled regions. However, the theory of Donnell and Wan does predict that as the thinness ratio of the shell is increased, greater imperfection sensitivity is experienced.

When a cylindrical shell is in a pure membrane state, the amount of energy stored in the shell can be quite large without a great deal of deformation. If the shell is in pure axial compression prior to buckling, the shell can fail (buckle) by developing a rapid exchange of energy from the membrane to the bending state. This is usually accompanied by a large amount of deformation. Prebuckling deformations due to imperfections cannot generally be seen by the naked eye, and yet they have a pronounced effect on the load-carrying capacity of the shell. These deformations, correctly accounted for by Donnell and Wan, result from nonlinear considerations. In the nonlinear case, Eq. (4) takes the form:

$$D\nabla^4 w + \frac{C}{R^2}\left(v_{,\theta} + w + \nu R u_{,x}\right) = N_x\left(w_{,xx} + \bar{w}_{,xx}\right) + \text{other nonlinear terms} \tag{5}$$

where

$$N_x = C[u_{,x} + \frac{1}{2}\beta_x^2) + \nu\left(\frac{v_{,\theta} + w}{R^2} + \frac{1}{2}\beta_\theta^2\right)]$$

With the success of the predicted behavior of a cylindrical shell subjected to an axial load, other attempts were made to examine the cylindrical shell subjected to other membrane states such as torsion (T) and hydrostatic pressure (p). The cylinder exhibited imperfection sensitivity for these loading conditions, but to a lesser degree than under axial load (P), especially in the postbuckled region as shown in Figure 2 in terms of axial displacement (D), twist, ϕ, and radial displacement W_R, respectively.

The radial displacement satisfying one solution of the nonlinear equation can be expressed in terms of a double Fourier series:

$$w = \sum_{i}^{\infty} \sum_{j}^{\infty} A_{ij} \cos i \frac{\pi x}{\ell_x} \cos j \frac{\pi y}{\ell_y} \tag{6}$$

The number of arbitrary parameters A_{ij} used in representing the postbuckled displacement pattern will dictate what minimum bifurcation load one can predict. The quantities ℓ_x and ℓ_y are the half wave lengths of the buckles in the axial and circumferential directions. The initial postbuckling state is represented by a checkerboard pattern (Figure 3), with λ being the load parameter and δ being the axial deflection. When internal pressure is considered, the solution can be greatly altered such that the final postbuckled state could be either diamond shaped (for low internal pressure) or ring shaped (for high internal pressure). Thus, three possible postbuckling patterns are possible at the critical limit point.

From these extremes in static behavior of shells when subjected to a compressive state, the concept of a classical bifurcation buckling can be developed. At any load parameter, Λ, below the lowest bifurcation load, the total energy of the system is increasing for any small perturbed load. If generalized coordinates q_i and q_j are selected such that they correspond to non-critical principal coordinates, then a test of the distinct critical paths can be made. When the energy is increasing with respect to any generalized coordinate, the shell is stable. This can be represented as though a ball is resting in a cup (Point A, Figure 4). As the state of stress approaches infinitesimally closer to the bifurcation load, a state of neutral equilibrium is

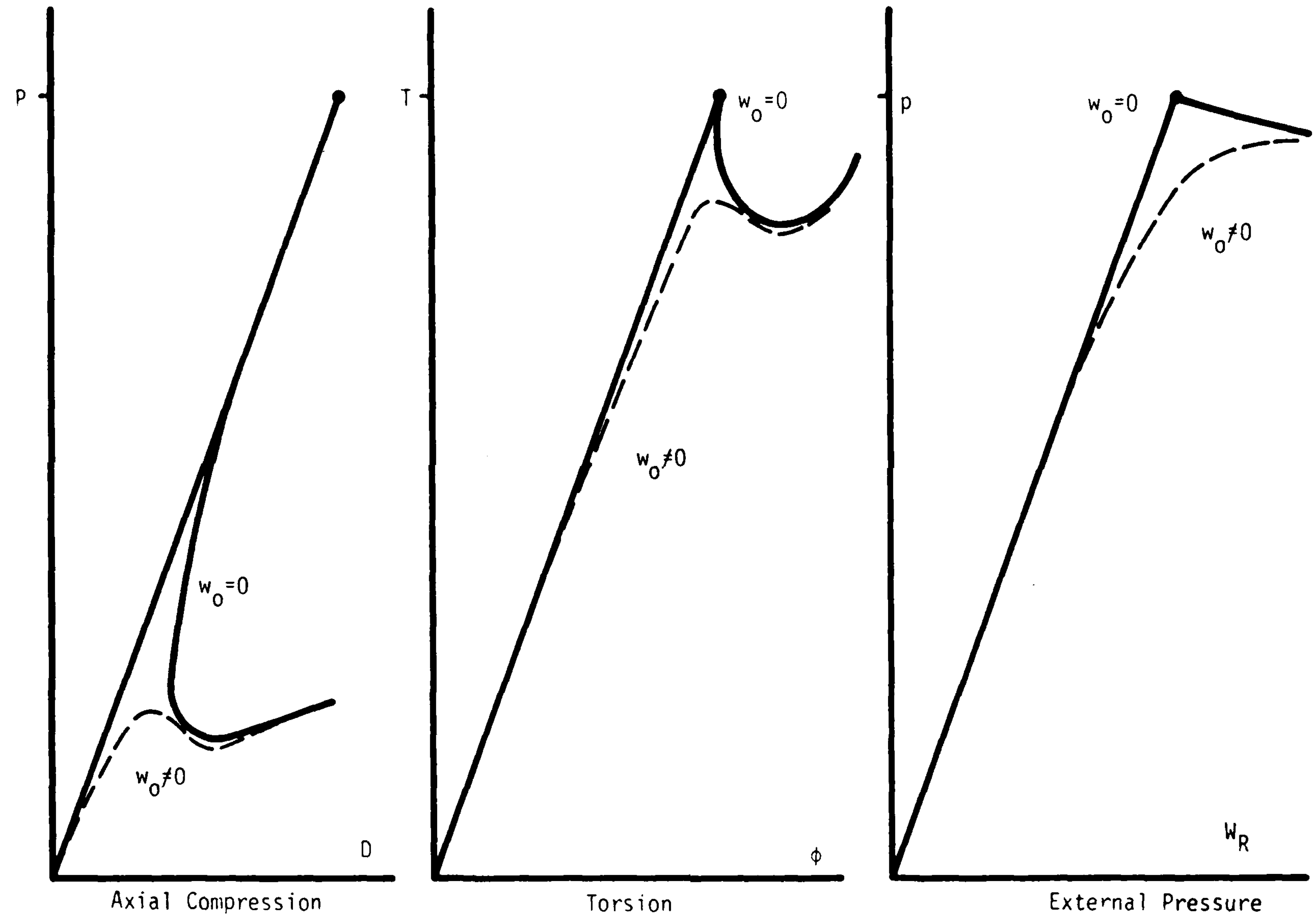


Figure 2. Typical postbuckling paths for cylinders.

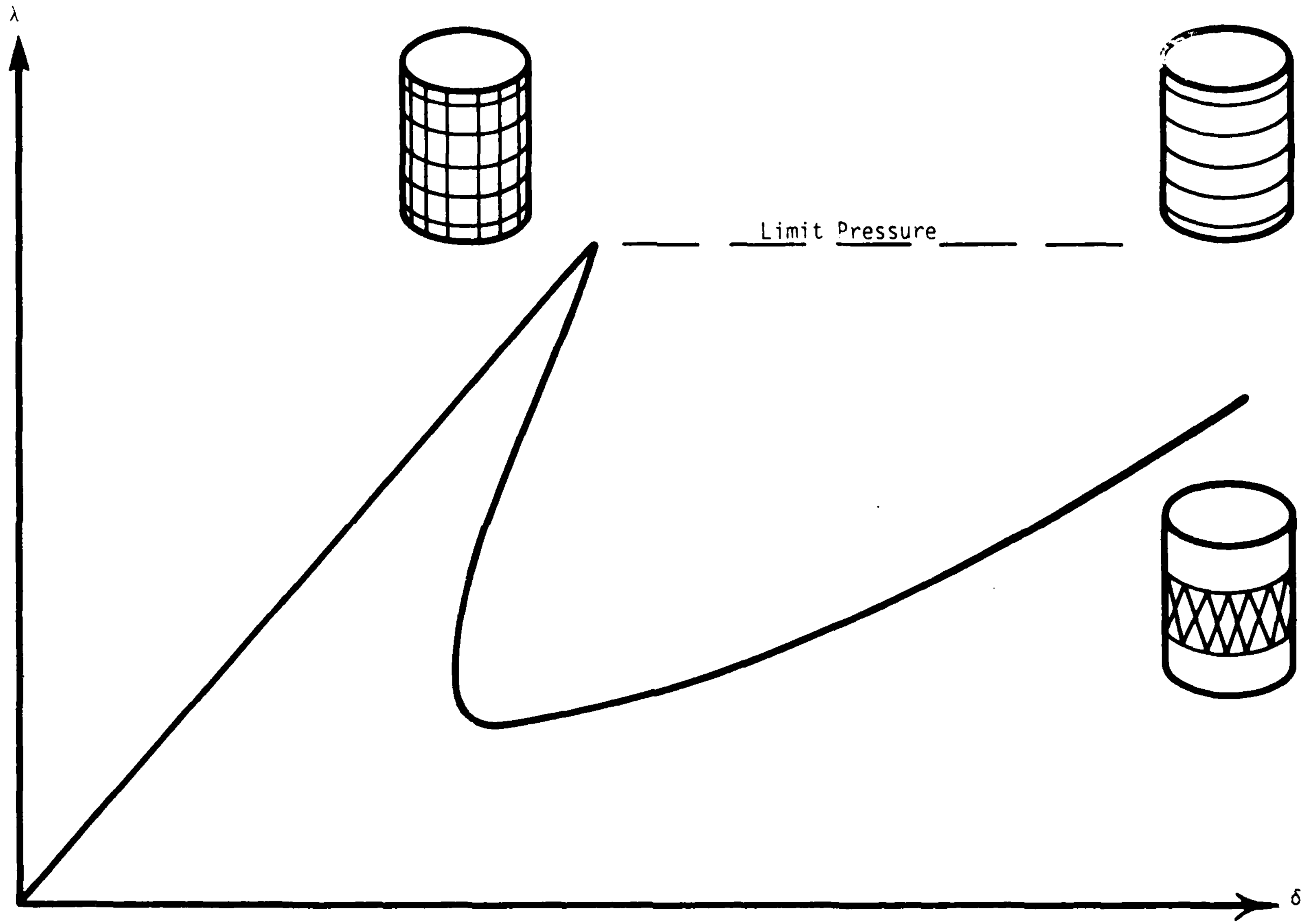


Figure 3. Response of cylinder to axial load.

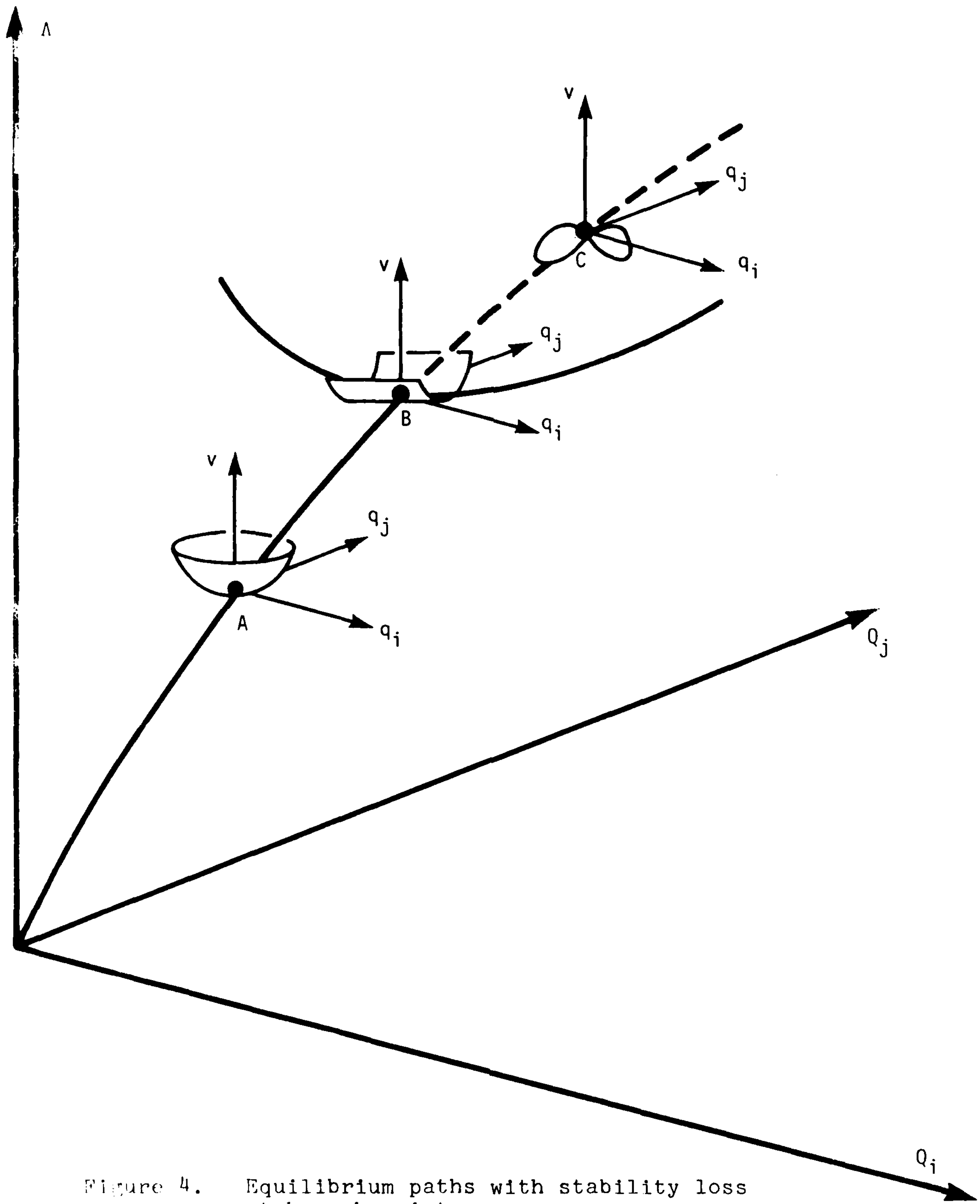


Figure 4. Equilibrium paths with stability loss at branch points.

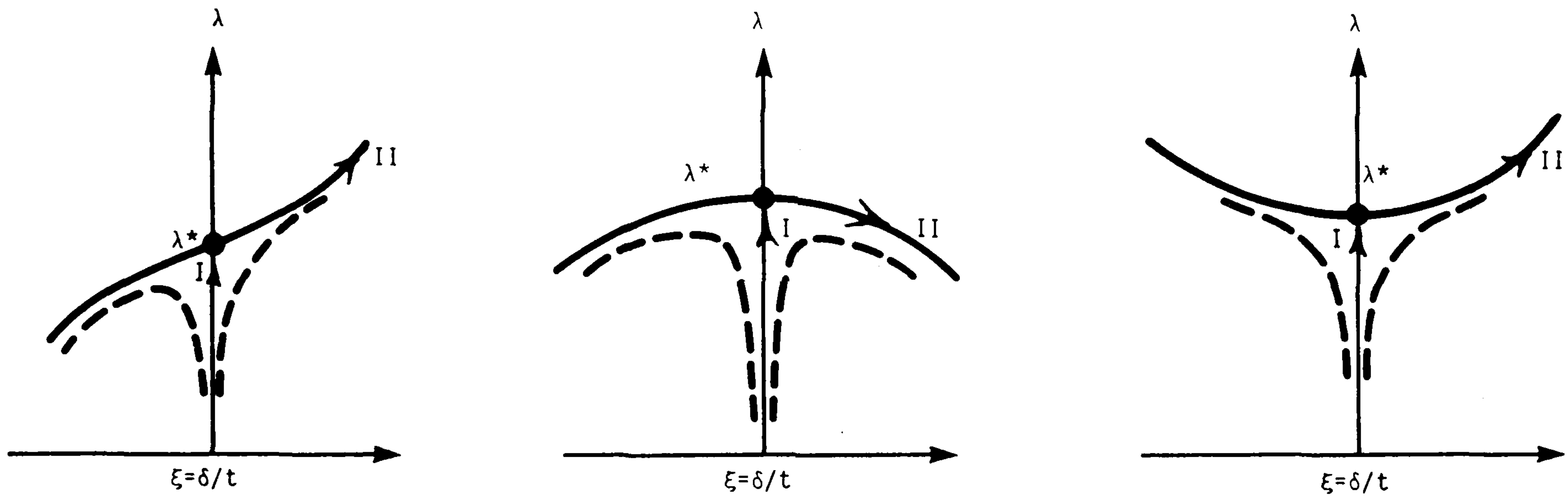
reached. Here, the energy is conditionally stable to some perturbed load and may be neutrally stable to a particular load. To determine the stability characteristics of the shell beyond this point, the energy must be traced for that perturbed load. This action is as though the ball were resting in a trough (Point B). Eventually any further increase in load will reach a peak (Point C) and ultimate collapse is reached.

When the generalized coordinate corresponds to the critical principal coordinate, postbuckling paths are possible (see Figures 5a-5c). The bifurcation process was considered to be a change from one equilibrium state of stress (path I) to an adjacent state of equilibrium (path II). When the critical classical buckling load (denoted as λ^*) is reached, infinitesimal adjacent states of equilibrium at the same load could exist.

The nonlinear postbuckling response of a totally enclosed liner or oil casing is represented in Figure 5a. The system is stable for one direction of motion and unstable in the opposite, thus unsymmetric postbuckling properties are possible. The response of a cylindrical shell subjected to axial compression or a spherical shell subjected to external pressure is represented by Figure 5b. The postbuckling behavior of a flat plate is represented by Figure 5c. For each case considered, the dotted line represents the behavior of an imperfect shell.

To determine the classical load λ^*, usually a set of ordinary differential equations are written and an eigenvalue of the system of equation is determined. The governing equations are derived from either equilibrium methods, energy methods, or imperfection methods.

Equilibrium methods can be used to examine the governing equations for a structure. Then, adjacent equilibrium states are hypothesized that satisfy the equations of equilibrium and the boundary conditions. These may be easily prescribed for simple systems such as columns but may be impossible to describe for complex shell structures.



(a) Unsymmetric - Unstable
$a \neq 0$, $b \neq 0$

(b) Symmetric - Unstable
$a=0$, $b<0$

(c) Symmetric - Stable
$a=0$, $b>0$

Figure 5. Possible bifurcation paths.

For energy methods, a deformation state is assumed to exist that satisfies the boundary conditions and continuity. By taking the variation of the potential energy with this assumed state and setting the results to zero, a condition of equilibrium is obtained. A load factor (eigenvalue) is determined when the second variation of the potential energy ceases to be positive definite.

Imperfection methods can be used to develop the equation of equilibrium for a system with imperfections. Eccentricity of load caused by the imperfection is examined, and a solution is obtained. The asymptotic character of the solution (usually radial displacement) is determined. When the deflections of the structure increase without bound for a relatively small increase in applied load, a load parameter is then said to be determined.

2.2 Semi-Empirical Methods

The fundamental approach or philosophy behind the employment of semi-empirical methods appears to be, in itself, quite straightforward. The classical buckling prediction for an idealized shell under idealized loads without the influence of boundary conditions can generally be easily determined. For example, the critical stress for the axially loaded cylinder can be expressed as:

$$\sigma_{cr} = \overline{C}\ Et/R \qquad (7)$$

where

$$\overline{C} = 1.0/\sqrt{3(1-\nu^2)}$$

or

$$\overline{C} = 0.605 \text{ for } \nu = 0.3$$

The quantity $\overline{C}$ is replaced by some functional quantity called a knockdown factor that accounts for a proper fit of all of the accepted experimental data. For example, Weingarten, et al.[319] suggests the relation:

$$\bar{C} = 0.606 - 0.546(1 - e^{\frac{-\sqrt{R/t}}{16}}) \qquad (8)$$

and shows that the above equation fits the lower bound of all the accepted data from fourteen different sources. The adequacy of the semi-empirical method is necessarily based upon a great deal of experimental data as well as some correlation of this data to a specific formula.

In 1957, Gerard and Becker[45,112-117] compiled a handbook of structural stability for NACA. Criteria were set for flat plates, shells, stiffeners, etc., which were based upon linear energy methods supplemented by semi-empirical theory. Batdorf's simplified methods[38-42] were used extensively in Gerard's comparison.

In 1968, under contract from NASA, Baker, et al.[31] collected various analysis procedures for shell structures encountered in the aerospace industry. Chapter 3 of this shell analysis manual cataloged a variety of stability criteria that was based upon the concept of semi-empirical methods. This work was later condensed and published by the original authors[32]. Similar collections were made by other organizations in the aerospace and energy fields. Unfortunately, much of this data was considered proprietary to the supporting organization.

With all of the complex problems associated with the determination of a stability criteria for stiffened and unstiffened shell structures, no single criterion can be used. Bounds on the application of the criteria need to be established. This was done to some extent by Baker, but it is not complete. An attempt to collect and computerize a semi-empirical stability criteria was made by Citerley[81]. Criteria from Baker[31], Smith and Spier[278] and Lakshmikantham and Gerard[187] was used. The only consideration of imperfection sensitivity was through a correlation factor. Usually the knockdown factor, $\bar{C}$ of Eq. (8), would incorporate this additional factor ($\gamma\bar{C}$). For example, the factor γ, given as:

$$\gamma = 1 - 0.901(1-e^{-\sqrt{R/t}\;/16}) \tag{9}$$

is suggested for curved isotropic panels. Buchert[60] applies the same basic idea to spherical, cylindrical shell-like structures and reticulated shell-like structures with positive and negative Gaussian curvature. Formulas are suggested that appear to have the same form as the classical critical stress prediction but considers an imperfection parameter, Δ, that includes not only the imperfect shape but also the deformation of the shell just prior to buckling. For example, the critical buckling stress for a spherical shell subjected to any loading is given by Eq. (7) where,

$$C = -\,0.54\,\frac{\Delta}{t_m} - 0.145\,H + \{1.09\,(\frac{\Delta}{t_m})^2 - 0.03\,\frac{\Delta}{t_m}\,H\}^{1/2} \tag{10}$$

$$+\;0.359\,(\frac{t_B}{t_m})^3\}^{1/2}$$

$$H = [9.9\,(\frac{\Delta}{t_m})^2 + 3.08\,(\frac{t_B}{t_m})^3]^{1/2}$$

t_m = effective membrane thickness

t_B = effective bending thickness

The cautions in adopting the knockdown factor concept are that one is willing to (1) accept the judgment of the original author as to what is credible data and (2) assure that the boundary conditions, loading and shell configurations are comparable.

Miller[205-207] has collected a great deal of information using empirical data and was the principal behind the criteria suggested for Code Case N-284 (Metal Containment Shell Buckling Methods) of the ASME.

At the same time that semi-empirical efforts were being advanced, more experimental results became available. Instead of establishing the precise behavior of shells, more questions were being raised. The effects of the testing machine compliance, probable boundary conditions and, in fact, even the buckled shape, were questioned. Yoshimura[334,335] showed that the cylindrical shell could be developed from a series of plane triangles, and thus the shell could buckle into a lower energy state (i.e., bending or folding along the intersection of the triangles). A crude mathematical development of the postbuckled state corresponds to the assumed three-term description suggested by von Karman. Ponsford[237] demonstrated that the buckled pattern was dynamic and could not be maintained even under controlled static loads. Almroth[3] found that by increasing the number of terms from 3 to 14, in representing the radial displacement and the initial imperfections, the value of the postbuckling load for the cylinder would decrease.

Further discussion of Donnell's approach was presented by Hoff[133,134] who illustrated that by evaluating over 1100 terms of the total potential energy, that in the limit the magnitude of the radial displacement had a significant effect on predicting the lower bound for the postbuckling state and that the Yoshimura pattern could be approached. This latter finding put a virtual end to the search of what was then the popular method of establishing a lower bound in the critical load through linear energy methods.

The effects of boundary conditions has also been reviewed[134,135,137,255,261,262,298]. Ohira[225], Nachbar and Hoff[215], Sobel[279], and Almroth[6] also discussed the problem of boundary conditions. Almroth developed relationships between the length of the shell and edge restraints for eight possible boundary conditions that could exist.

2.3 Geometric Nonlinear Theories

Continued research in the adaptation of von Karman's nonlinear shell theory was made by Leggett and Jones[191] and by Michielsen[203]. Unfortunately, little information has been published on the nonlinear methods. Chien[78] presented one theory that was later criticized by Goldenveizer and Lur'ye[119] and Reiss[239]. The major complaint was that Chien's development could not handle displacement boundary conditions correctly. Reissner[241] developed finite displacement equations of equilibrium for the axisymmetric response of a shell of revolution. This investigation was far-reaching inasmuch as the extension of linear methods had been shown.

The development of the nonlinear equations of equilibrium for general shells has had less intense study than the linear set. From a general point of view, the discussion of Zerna[336] gives the most complete theory on the nonlinear behavior. Naghdi[217] has also presented a nonlinear theory. Perhaps the most commonly accepted theory is that of Sanders[254], which is limited to small strains and moderately small rotations. Reissner[242] developed a system of nonlinear differential equations for the symmetric deformation of shells permitting large rotations and finite strains. Danielson[95] developed the bifurcation and postbuckling equations using asymmetric expansion techniques from an energy criterion. Yokoo and Matsunaga[332] derived a fundamental set of two-dimensional shell equations. The structures under investigation were made of rather simple geometric forms, although some investigation was made of composite vessels, such as the joining of cylinders and hemispherical heads[127].

3.0 RECENT DEVELOPMENTS IN ANALYTICAL METHODS

As stated previously, the analytical methods used in determining the imperfection sensitivity of shells usually follow three avenues:

(1) Equilibrium methods
(2) Energy methods
(3) Imperfection methods

Each of these methods has been able to demonstrate that the critical load of a particular shell subjected to a given load can be reasonably predicted. For many engineers, these predictions are too cumbersome and a simple semi-empirical knockdown factor would do. However, when the conditions under investigation go outside established limits, the engineer should exercise caution. Further, the engineer must assess the difference between the real world and the highly idealized buckling analysis. In many instances these differences are collectively classified as imperfections. Some areas that have been separately identified are:

(1) Geometric imperfections
(2) Boundary conditions
(3) Stiffness or material property variation
(4) Nonuniformity of load
(5) Prebuckling deformation
(6) Dynamic loads and responses
(7) Nonlinear material behavior

In order to act intelligently, the engineer must be aware of the accepted analytical methods and their possible overlapping range of applicability and controversy. To do this, the details of each method must be examined. Only the first two methods will be discussed in detail here. Because of their complexity, only the highlights will be presented. The derivations are exceedingly difficult and generally require the introduction of some mathematical hypothesis that may not be of interest to the casual reader. Therefore, this presentation will only attempt to demonstrate the complexities associated with the applicable method, and will not provide a complete and sufficient derivation of each method. The cylinder under axial compression has been used as the classic problem to demonstrate the conditions of stability and the effects of imperfection. The discussion will therefore be limited to this case.

3.1 Unstiffened Shells

3.1.1 Equilibrium Method

In order to amplify the above statements, and to gain a more thorough understanding of the buckling branching paths, consider a perfect cylinder with a radius, R, thickness, t, and length, L. Assuming a radial displacement, w, in a shallow shell Cartesian system [x, y or Rθ] and with the use of the Airy stress function, Φ, the membrane stress resultants are defined by:

$$\begin{aligned} N_x &= \Phi_{,yy} \\ N_y &= \Phi_{,xx} \\ N_{xy} &= \Phi_{,xy} \end{aligned} \tag{11}$$

Donnell's equilibrium equations are then defined in terms of the Airy's function and a displacement function, W:

$$\frac{1}{Et} \nabla^4 \Phi - \frac{1}{R} W_{,xx} + \frac{1}{2} L(W,W) = 0 \tag{12a}$$

$$\frac{Et^3}{12(1-\nu^2)} \nabla^4 W + \frac{1}{R} \Phi_{,xx} - L(\Phi,W) = 0 \tag{12b}$$

Where ∇^4 is the biharmonic operator $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})^2$ and the operator L(Q,V) is given by:

$$L(Q,V) = Q_{,xx} V_{,yy} - 2Q_{,xy} V_{,xy} + Q_{,yy} V_{,xx} \tag{13}$$

For a possible three mode postbuckled response denoted as (ξ_1, ξ_2 and ξ_3 -- one axisymmetric and two non-axisymmetric) with a cylinder subjected to an axial stress, σ, the displacement can be assumed as:

$$w = \frac{\nu\sigma R}{E} + W \tag{14a}$$

and Airy's function as:

$$\Phi = -\frac{1}{2}\sigma t y^2 + \phi \tag{14b}$$

The particular solution form of the functions ϕ is assumed to be a double Fourier series such as Eq. (6). The more general solution considers sine terms as well. This particular solution is substituted into Eq. (12a) and the coefficients are determined by equating like terms. Substituting a like function in W also including sine terms, if applicable, into Eq. (12b) and applying a Galerkin procedure, yields an error function. This function in turn is used along with orthogonality conditions of the assumed series to yield the relationship:

$$\begin{aligned} &\xi_1(a_1-\lambda) + (\tfrac{1}{2}a_2+a_3)(\xi_2^2-\xi_3^2) = 0 \\ &\xi_2[a_4-\lambda + 8(a_2+2a_3)\ \xi_1] = 0 \\ &\xi_3[a_4-\lambda - 8(a_2+2a_3)\ \xi_1] = 0 \end{aligned} \tag{15}$$

Equation (15) identifies the possible branching paths that a perfect shell can take when multi-modes exist as λ approaches λ^* (see Figures 6a and 6b).

The bifurcation of a shell can be reached under two conditions: $a_1 > a_4$ or $a_1 < a_4$. For the first case ($a_1 > a_4$), the linear path I is followed until $\lambda=\lambda^* = a_4$. Then, depending upon the participation of the postbuckling modes, four possible distinct paths can be followed:

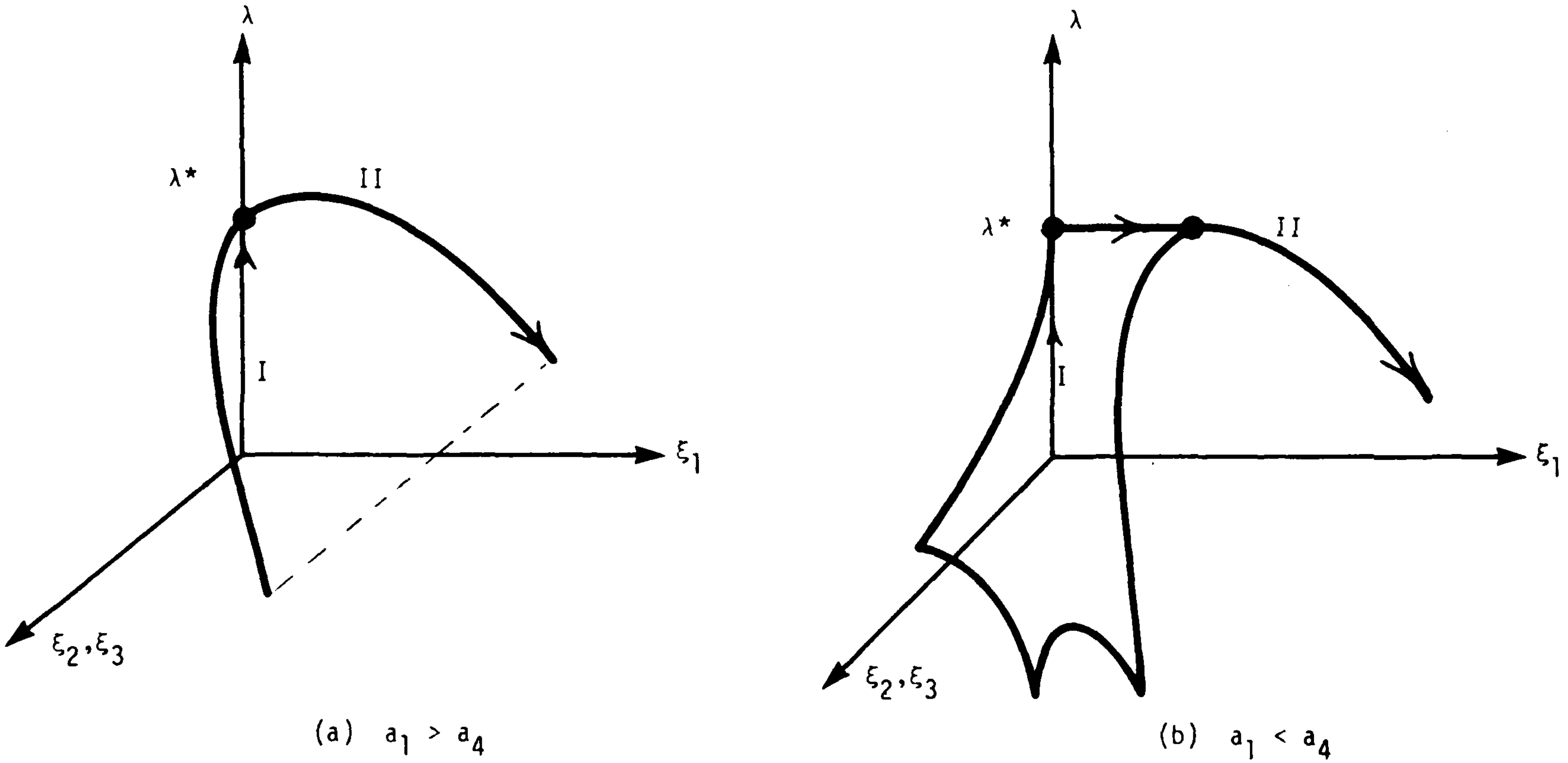


Figure 6. Two-mode branching.

$\xi_2 = \xi_3 = 0$ and $\xi_1 \neq 0$

ξ_2 or $\xi_3 \neq 0$ and $\xi_1 = 0$

ξ_2 or $\xi_3 \neq 0$ and $\xi_1 \neq 0$

ξ_2 and/or $\xi_3 \neq 0$ and $\xi_1 \neq 0$

These branching paths can be approached from either the positive or negative side of each mode. When $a_1 < a_4$, the paths are slightly more restrictive. First, path I is followed until $\lambda = \lambda^* = a_1$. Branching can then take place when:

$$\xi_2 = \xi_3 = 0 \text{ and } \xi_1 \neq 0 \text{ until } \xi_1 = \frac{a_4 - a_1}{8(a_2 + 2a_3)}$$

then bifurcation continues into the ξ_2 mode. A similar response in the ξ_3 mode can take place when $\xi_1 < 0$. For all of these cases an unstable buckling path is followed.

The particular form of the coefficients a_1 thru a_4 are dependent upon the form of the function w in Eq.(14) Since a sine or cosine function is a solution to Eq. (12) w usually is also assumed to have the same form. If an imperfection $\bar{w}$ is assumed to exist with the same content as w, then the operators of Eq. (12) would be $\frac{1}{2}L(w, w+2\bar{w})$ and $-L(\phi, w+\bar{w})$ respectively.

Arbocz[25] gives a rather complete description of the above procedure and includes the effects of an initial imperfection and a three term radial response. Only two simultaneous modes of the three were considered in the analysis. Later, Imbert[156] examined the three mode responses and demonstrated how the coefficients of Eq. (15) relate to the imperfection unevenness parameters expressed by Donnell and Wan. Both investigations amply demonstrate the mathematical complexities they encounter when pursuing this type of analysis. The fruits of their effort can be utilized in an efficient computerized procedure. However, the whole analysis process becomes very complicated if one tries to include higher order terms of the Fourier series.

3.1.2 Koiter's Method

Koiter's thesis on the stability of elastic equilibrium in 1945[169] is perhaps the most significant piece of work done in that decade. Unfortunately, his work was not available in English until 1967[175]. The original work was performed between 1940 and 1943 but because of World War II his work was isolated from the rest of the engineering community[180]. Koiter formulated the basis of the initial postbuckling theory from energy considerations which was triggered by a hypothesis suggested by Cox[91].

Koiter's statement of elastic stability assumes the existence of a potential energy functional P[u] where u is the displacement state which excludes inelastic behavior. Two adjacent states are assumed to exist at the bifurcation load λ^*. The potential energy functional is expanded into a series of like terms:

$$P^{\lambda}[u] = P_1^{\lambda}[u] + P_2^{\lambda}[u] + P_3^{\lambda}[u] + P_4^{\lambda}[u] + \ldots + \varepsilon Q_1^{\lambda}[u] \geq 0 \quad (16)$$

where the notation $P_m^{\lambda}[u]$ is the collection of like powers of mth order of u and its derivatives for a load, λ, and $Q_1^{\lambda}[u]$ includes the effects of the initial imperfection and ε denotes the magnitude. By first restricting the analysis in the neighborhood of the bifurcation point, Koiter assumed that the displacement field immediately adjacent to the critical bifurcation point is essentially of the same form as the classical buckling modes. From variational principles, it is well known that first variation of the potential energy is a necessary and sufficient condition to prove equilibrium. Koiter shows that by expanding the potential energy function for a linear combination of adjacent buckling modes, the system is stable when the second variation of the potential energy is semi-definite positive. Further the third variation, $P_3[u]=0$ and/or the fourth variation $P_4[u]>0$ for all the buckling modes. The postbuckling behavior of shells as represented in Figure 5 was suggested by Koiter. Once the basic idea had been presented, several extensions of the method were presented by Koiter[170-177] and others. Seide[257] presents a rather condensed, yet complete, description of Koiter's theory which is applicable to shells. Budiansky and Hutchinson and their colleagues at Harvard have contributed to the understanding of Koiter's approach and have successfully advanced the method in several areas[61-65,147-153].

In order to demonstrate the usefulness of Koiter's simple yet elegant approach, Budiansky[63] in 1967 gave a presentation on the theory of initial postbuckling behavior of shells. He examined the asymptotic behavior of the solution to the governing equation of a cylinder subjected to torsion. Amazigo and Budiansky[9] perhaps have described the more general approach, while Stephens[283] applied the procedure to a cylindrical panel, and Fitch[105] applied the method to spherical caps. The Koiter analysis is carried out by assuming that the deflection and Airy's stress function can be expanded into the form:

$$\begin{aligned} W &= W_o + \xi W_1 + \xi^2 W_2 + --- \\ \Phi &= \Phi_o + \xi \Phi_1 + \xi^2 \Phi_2 + --- \end{aligned} \tag{17}$$

where the subscript o refers to the prebuckled state, 1 and 2 are classical orthogonal buckling modes and ξ represents the magnitude of each mode. The load factor λ can be expanded in terms of powers of ξ :

$$\lambda/\lambda^* = 1 + a\xi + b\xi^2 + --- \tag{18}$$

where λ^* is the bifurcation load.

The procedure for determining the coefficients a & b in Eq. (18) has been outlined by Budiansky[64]. For the symmetric postbuckling paths (a=0), the parameter b for a cylinder can be defined as:

$$
\begin{aligned}
b = c \iint \Big[2 \, \{ & \Phi_1^{\bullet\bullet} W_1' \, W_2' + \Phi_1'' W_1^{\bullet} \, W_2^{\bullet} \\
& - \Phi_1^{\bullet\prime} (W_1' W_2^{\bullet} + W_1^{\bullet} W_2') \} + \Phi_2'' (W_1')^2 \\
& + \Phi_2^{\bullet\bullet} (W_1^{\bullet})^2 - 2 \Phi_2^{\bullet\prime} W_1' W_1^{\bullet} \Big] \, dxdy \Big/ \iint \sigma_{cr} (W_1^{\bullet})^2 \, dxdy
\end{aligned}
\tag{19}
$$

where

$(\ )^{\bullet} = \dfrac{\partial}{\partial x}$

$(\ )' = \dfrac{\partial}{\partial y}$

σ_{cr} = critical buckling stress

$c = \sqrt{3(1-\nu^2)}$

Parameter b (Koiter's parameter) indicates the postbuckling character of the shell. When b is negative, path II of Figure 5b is followed (unstable). When b is positive, path II of Figure 5c is followed (stable).

When a shape imperfection ($\overline{\xi} = w_o/t$) is introduced, the behavior of the shell depends upon the amplitude of the imperfection. The buckling strength of the axially compressed circular cylinder is greatly reduced by relatively small imperfections. Koiter has shown that the critical load factor λ is governed by:

$$(1-\lambda/\lambda^*)^{3/2} = \frac{3\sqrt{3}}{2} \sqrt{-b} \quad (\lambda/\lambda^*) \; |\overline{\xi}| \tag{20}$$

when $a = o$ and $b < o$, as in Figure 5b. The negative value of the factor b causes a reduction in the buckling load, and is depicted by the dotted line in Figure 5b for the imperfect shell. A restrictive assumption implied in the derivation of Eq. (20) is that $\xi \leq 1$. No attempt should be made to apply this result beyond this limit. Koiter shows that an axisymmetric imperfection can cause a large reduction in the load-carrying capacity of a cylinder, as indicated in Figure 7, and he also concludes that asymmetric imperfections can have an even more pronounced effect. From the collection of data from many tests, a knockdown factor

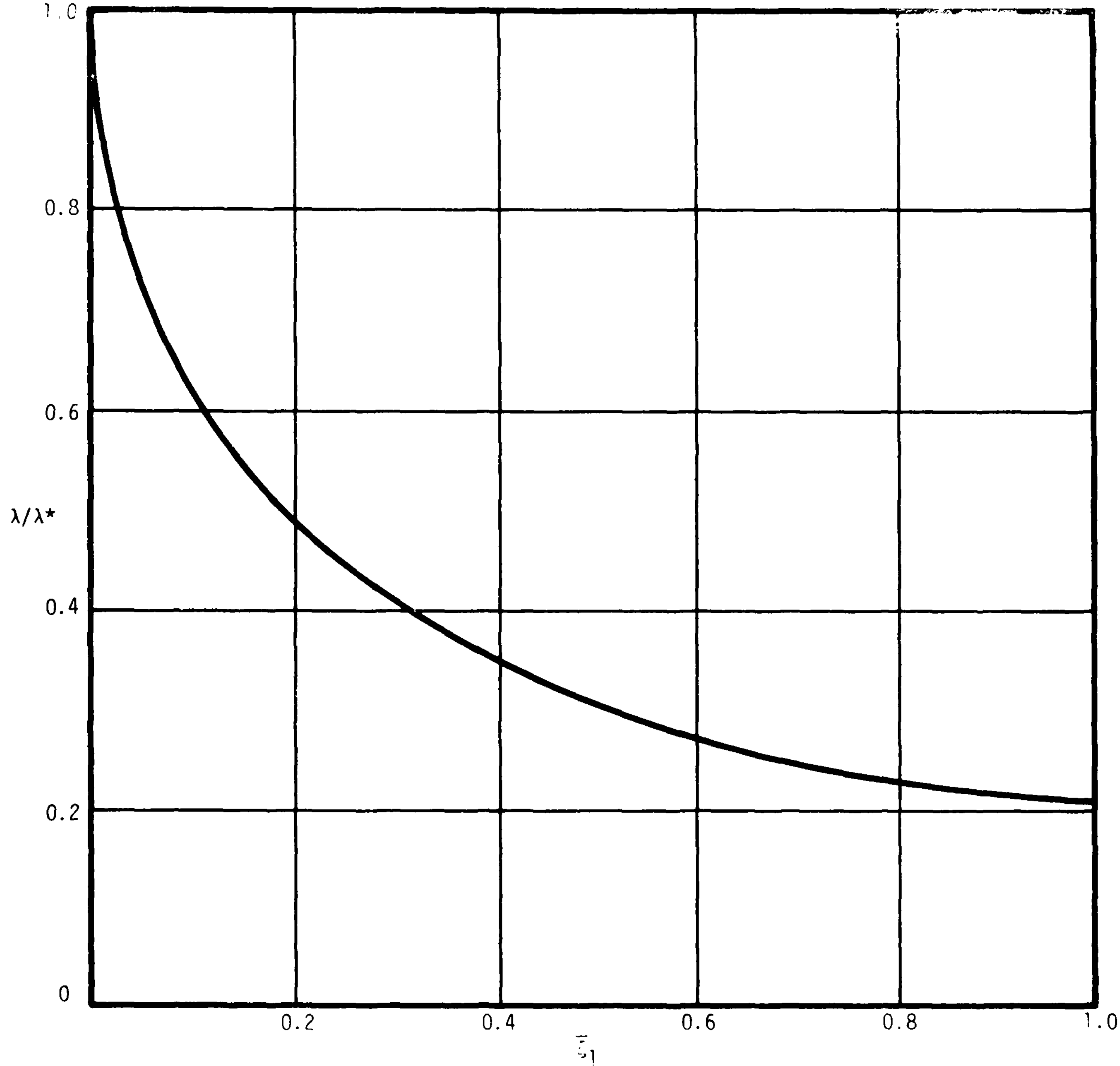


Figure 7. Effect of axisymmetric imperfection on buckling of cylindrical shell.

between 0.25 and 0.33 has been suggested for the axially loaded cylinder. Koiter suggests that due to axisymmetric imperfections, a knockdown factor of 0.10 can be used as the upper bound for large imperfections. However, Seide re-examined Koiter's theory and demonstrated that the Koiter analysis scheme is quite limited and may not accurately describe the failure mode.

Most of the investigators who have used Koiter's theory on different shells of revolution have used the buckled mode shape as the assumed imperfection shape. Lange and Newell[188] have suggested that certain imperfection patterns are found on spheres (HEXAGONS) and cylinders (DIAMONDS), and Hutchinson[150] has demonstrated that for a spherical shell having a radius, R, and thickness, t, the imperfection shape could take the form:

$$w_o = \bar{\xi}_1 \; t \cos\left(q_o \frac{x}{R}\right) + \bar{\xi}_2 \; t \sin\left(q_o \frac{x}{2R}\right) \sin\left(q_o \frac{\sqrt{3}}{2} \frac{y}{R}\right) \tag{21}$$

where

$$q_o^4 = 12(1-\nu^2) \left(\frac{R}{t}\right)^2$$

This led to a buckling prediction equation of critical (p)/classical (p*) pressure ratios:

$$\begin{aligned} (1 - p/p^*)^2 &= \frac{27\sqrt{3}}{32} \left|\bar{\xi}_2\right| p/p^* \quad ; \quad \bar{\xi}_1 = 0, \quad \bar{\xi}_2 \neq 0 \\ (1 - p/p^*)^2 &= \frac{9\sqrt{3}(1-\nu^2)}{8} \bar{\xi}_1 \; p/p^* \quad ; \quad \bar{\xi}_1 > 0, \quad \bar{\xi}_2 = 0 \end{aligned} \tag{22}$$

These same equations were verified by Reissner[243] using certain orthogonality relations in Koiter's procedure "as a consequence of the nonappearance of secular terms." Citerley, et al.[82], has demonstrated through imperfection methods that when the mode shape is comprised of an axisymmetric <u>and</u> non-axisymmetric imperfections ($\bar{\xi}_1 > 0$; $\bar{\xi}_2 > 0$) a further reduction is realized. Moreover, the influence of a boundary interacts with any shape imperfection to reduce the buckling load.

Since shell structures are fabricated, they are subject to manufacturing tolerances. In an attempt to predict the buckling behavior of these shells, it must be assumed that the imperfections occur in a random manner[11]. For cylinders, Fersht[104] found that axisymmetric random imperfections reduced the buckling load less than nonaxisymmetric random imperfections. Other imperfection geometries have been studied to determine what local effects of assumed shapes have on the overall stability characteristics of a shell. Some of the most interesting comparisons with experimental results in this area are those presented by Tennyson and his associate Muggeridge[213], Caswell,Chan[291-296] and Hutchinson[154].

Finally, Arbocz, in his thesis[22], demonstrated than an adequate prediction of the knockdown factors can be performed if the shape of the imperfection is accurately known. In subsequent analyses the effects of boundary conditions were studied[27]. Some attempts to store the imperfection data of test specimens into a data bank have also been implemented[28].

3.1.3 Thompson's Extension of Koiter's Method

One group that has developed a comprehensive understanding of Koiter's principles is at University College, London, and consists of Thompson and Hunt[299-302], Roorda[246] and Supple[287-288]. It has been demonstrated by Roorda that certain structures could behave in a manner that is not necessarily predicted either by the equilibrium or the imperfection methods. Thompson and Hunt[300] have given a rather complete mathematical description of implementing conceptually the energy theorems to an optimum structure.

Supple[287] illustates the basic concept of Thompson's generalized coordinates in the general theory of elastic stability through an example of symmetric structural systems. The potential energy, P, of a structural system is assumed to be a function of a generalized load parameter f and two non-dimensional generalized coordinates, u_1 and u_2. The latter two represent possible buckling modes. For an incremental change in load, δf, from an inital value, f_o, the potential energy can be written in the form:

$$P(f,u_i) = P(f_o + \delta f, u_i) \tag{23}$$

Expanding the right hand side in the form of a Taylor series:

$$\begin{aligned} P(f,u_i) = P(f_o,0) &+ \left.\frac{\partial P}{\partial u_i}\right|_{f_o} u_i + \left.\frac{\partial P}{\partial f}\right|_{f_o} \delta f \\ &+ \frac{1}{2!}\left[\left.\frac{\partial^2 P}{\partial u_i \partial u_j}\right|_{f_o} u_i u_j + 2\left.\frac{\partial^2 P}{\partial u_i \partial f}\right|_{f_o} u_i \delta f\right] \\ &+ \frac{1}{3!}\left[\left.\frac{\partial^3 P}{\partial u_i \partial u_{i_j} \partial u_k}\right|_{f_o} u_i u_j u_k + 3\left.\frac{\partial^3 P}{\partial u_i \partial u_j \partial f}\right|_{f_o} u_i u_j \delta f\right] \end{aligned} \tag{24}$$

where the subscripts i,j,k are able to take the value of 1 and 2.

If the buckling modes u_1 and u_2 are assumed to be small, the two equilibrium equations for a symmetric system take the form:

$$\begin{aligned} \frac{\partial^2 P}{\partial u_1^2} u_1 + \frac{1}{3!}\left(\frac{\partial^4 P}{\partial u_1^4} u_1^3 + 3\frac{\partial^4 P}{\partial u_2^2 \partial u_2^2} u_1 u_2^2\right) + \frac{\partial^3 P}{\partial u_1^2 \partial f} u_1 \delta f = 0 \\ \frac{\partial^2 P}{\partial u_2^2} u_2 + \frac{1}{3!}\left(\frac{\partial^4 P}{\partial u_2^4} u_2^3 + 3\frac{\partial^4 P}{\partial u_1^2 \partial u_2^2} u_1^2 u_2\right) + \frac{\partial^3 P}{\partial u_2^2 \partial f} u_2 \delta f = 0 \end{aligned} \tag{25}$$

Three solutions exist for a doubly symmetric system. Two solutions referred to as uncoupled modal solutions are given by:

$$u_1^2 = -\frac{3!(\delta f_1)\dfrac{\partial^3 P}{\partial u_1^2 \partial f}}{\dfrac{\partial^4 P}{\partial u_1^4}} \quad ; \qquad u_2 = 0 \tag{26}$$

$$u_2^2 = -\frac{3!(\delta f_2)\dfrac{\partial^3 P}{\partial u_2^3 \partial f}}{\dfrac{\partial^4 P}{\partial u_2^4}} \quad ; \qquad u_1 = 0 \tag{27}$$

where δf_1 and δf_2 are different critical load values; i.e., at $u_1 = u_2 = 0$, $f_1 = f_o + \delta f_1$, $f_2 = f_o + \delta f_2$ corresponding to the different buckling modes of u_1 and u_2. The quadratic character of these solutions follows the individual bifurcation paths in Figures 5b and 8a for $\partial^4 P/\partial u_i^4 < 0$, and Figures 5c and 8b for $\partial^4 P/\partial u_i^4 > 0$. Since i represents two possible modes, two additional combinations can exist:

$$\partial^4 P/\partial u_1^4 < 0 \quad ; \quad \partial^4 P/\partial u_2^4 > 0 \text{ and } \partial^4 P/\partial u_1^4 > 0 \quad ; \quad \partial^4 P/\partial u_2^4 < 0.$$

These are illustrated in Figure 8c.

The third solution is for the coupled mode, which is satisfied by:

$$\begin{aligned} &\left(\frac{\partial^3 P}{\partial u_2^2\ \partial f} \cdot \frac{\partial^4 P}{\partial u_1^4} - 3\ \frac{\partial^3 P}{\partial u_1^2 \partial f} \cdot \frac{\partial^4 P}{\partial u_1^2 \partial u_2^2}\right) u_1^2 \\ &+ \left(3\ \frac{\partial^3 P}{\partial u_2^2 \partial f} \cdot \frac{\partial^4 P}{\partial u_1^2 \partial u_2^2} - \frac{\partial^3 P}{\partial u_1^2 \partial f} \cdot \frac{\partial^4 P}{\partial u_2^4}\right) u_2^2 \\ &= -6\ \frac{\partial^3 P}{\partial u_1^2 \partial f} \cdot \frac{\partial^3 P}{\partial u_2^2 \partial f}\ \Delta f \end{aligned} \tag{28a}$$

where $\Delta f = f_1 - f_2$

Four possible bifurcation conditions can exist for the coupled modes:

Condition 1: A transition path between the uncoupled postbuckling equilibrium paths.

Condition 2: Always from the primary uncoupled postbuckling path.

Condition 3: Always from the secondary uncoupled postbuckling path.

Condition 4: From one or both of the uncoupled modes.

When both uncoupled modes are rising (Figure 8a), the coupled buckling path from the primary equilibrium paths will be rising (symmetric-stable) when:

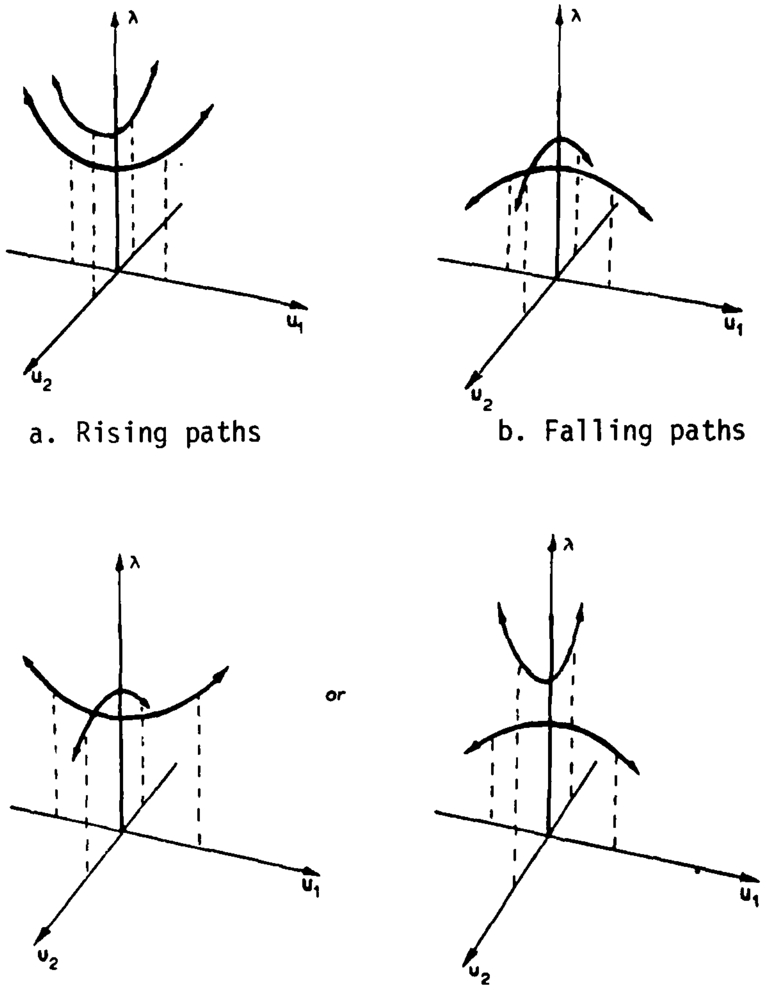
$$\frac{\partial^4 P}{\partial u_1^4} \cdot \frac{\partial^4 P}{\partial u_2^4} > \left(3 \frac{\partial^4 P}{\partial u_1^2 \partial u_2^2}\right)^2 \tag{28b}$$

and will be falling (symmetric-unstable) when:

$$\frac{\partial^4 P}{\partial u_1^4} \cdot \frac{\partial^4 P}{\partial u_2^4} < \left(3 \frac{\partial^4 P}{\partial u_1^2 \partial u_2^2}\right)^2 \tag{28c}$$

When either or both uncoupled postbuckling paths are unstable, the coupled buckling paths branching from the primary uncoupled path will always be unstable. When one or both uncoupled paths are stable, the coupled path from the secondary path will always be stable. When both modes are unstable (Figure 8b), branching from the secondary path will be stable when Eq. (28b) is met, and unstable when Eq. (28c) is met. The complexities of evaluating the stability characteristics of a shell when employing this extension of the Koiter method begins to overshadow the final results.

In Koiter's method, a single parameter, b, defined by Eq. (19), is used for the determination of a stable shell. In Thompson's approach, rather than developing a single parameter, a more general approach is used. Through the employment of calculus of variation principles, different conditions of the total strain energy can be examined. It appears that Thompson's extension of Koiter's method not only gives the same results as the original, but also predicts some additional behavior. A comparison of Fig. 6a and Figure 9iii suggests the similarity of the two methods.



a. Rising paths

b. Falling paths

c. Mixed -- falling and rising paths

Figure 8. Uncoupled branching configurations for doubly-symmetric structural systems [287].

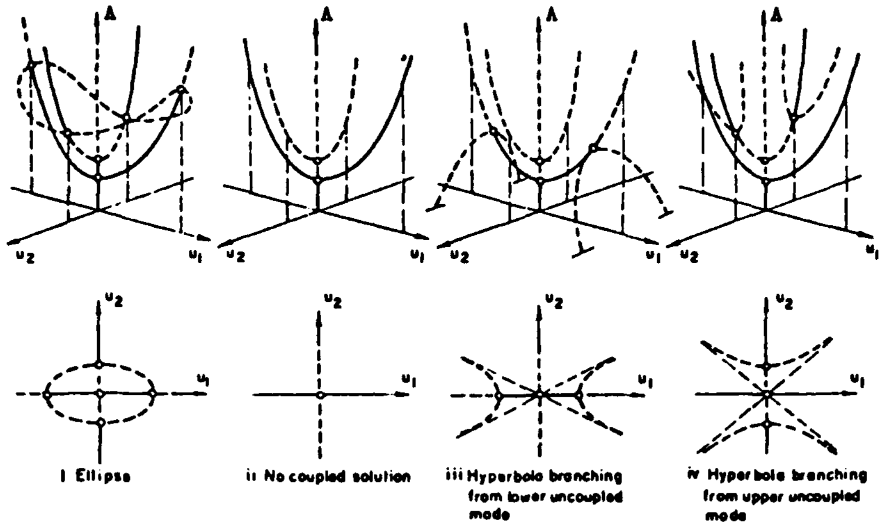


Figure 9. Forms of coupled postbuckling for ideal doubly-symmetric structural systems (—— stable equilibrium path; ---- unstable equilibrium path) [287].

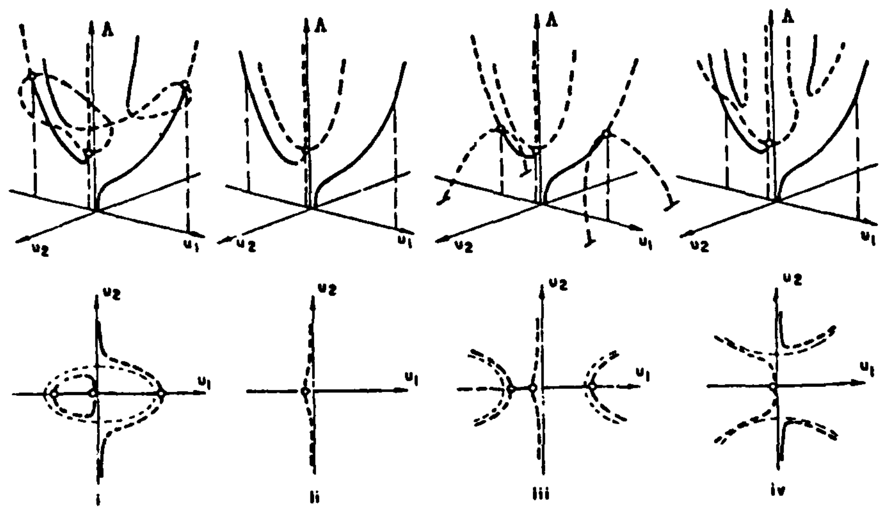


Figure 10. Forms of postbuckling equilibrium path for doubly-symmetric structural systems with imperfections ε (—— stable equilibrium path; - - unstable equilibrium path) [287].

The important finding from Thompson's approach is that even with two uncoupled buckling modes, which are on stable rising paths, unstable coupled paths are possible. This phenomena would have been ignored by other methods. Similar conclusions are drawn for systems with imperfections (Figure 10).

In a more abstract approach, Thompson[302] has utilized catastrophe theory in order to establish the precise number and type of imperfections to which a system will be most sensitive. He suggests that instead of a two mode interaction representation of imperfections, seven control parameters may be required; one of which is the load. Thus, six control parameters are then required. The surface that represents one of the fundamental control parameters, a hyberbolic umbilic, as described by Hunt[301], is illustrated in Figure 11. Although this approach has not been thoroughly tested or accepted, it may suggest why proponents of one or another type of imperfection parameter may be partially correct in their findings.

Although the details may be too idealized for many, two quite simple theorems are postulated by Thompson and Hunt that tend to lay a foundation for the understanding of bifurcation in shell structures and the mechanisms that induce the failure:

"Theorem 1: An initially stable (primary) equilibrium path rising monotonically with the loading parameter cannot become unstable without intersecting a further distinct (secondary) equilibrium path.

"Theorem 2: An initially stable equilibrium path rising with the loading parameter cannot approach an unstable equilibrium state from which the system would exhibit a finite dynamic snap without the approach of an equilibrium path (which may or may not be an extension of the original path) at values of the loading parameter less than that of the unstable state."

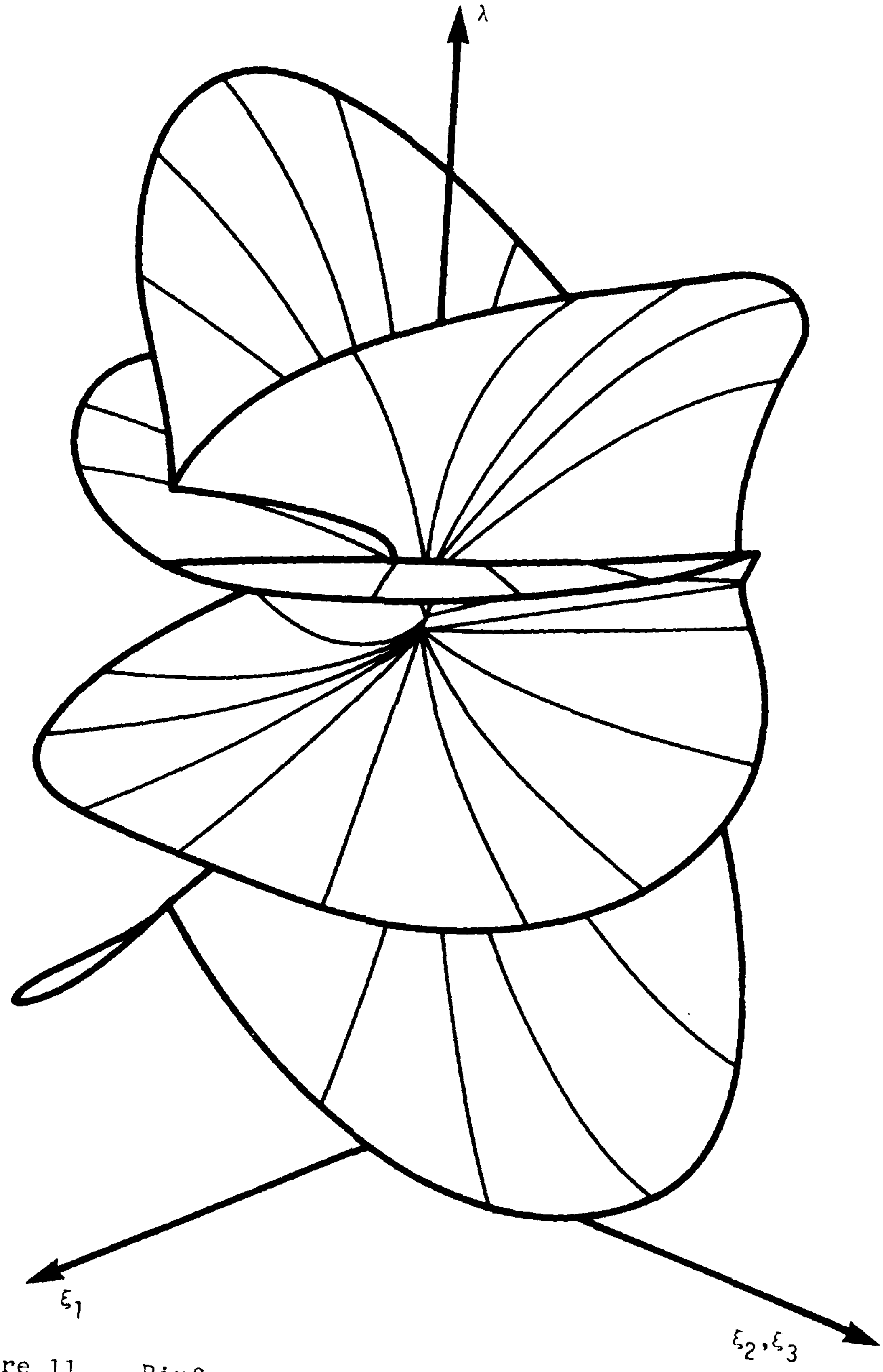


Figure 11. Birfurcation paths based upon catastrophe theory.

These theorems would appear to be a rather complicated statement for such a simple mechanism of single path bifurcation. On the other hand, for the more general case, complex functionals may be required to represent the total energy. This means that more complicated variational methods need be applied to make a proper evaluation. However, with these theorems in mind, a clearer understanding is developed on the possible paths or branching mechanisms of shell buckling into symmetric and asymmetric, stable and unstable postbuckling modes. This type of evaluation can best be done using energy methods. Since it is a nonlinear problem, the solution to a specific set of conditions is not unique and is path-dependent. The path that the solution follows has been shown to be influenced by imperfections. Energy methods can examine the overall behavior of a shell's buckling characteristics and establish the final equilibrium position (if one exists).

3.2 Imperfection Sensitivity of Stiffened Shells

A great deal of effort has been expended in recent years in an attempt to more fully understand the instability of stiffened shells. Examples are Amazigo[8,10], Brodsky[55,56], Bronowicki[57], Card[75], Goldberg[118], Heard[130], Khot[163,164], Lee[190], McGinley[201], Pedersen[229], Singer[271], Tvergaard[307] and Viswanathan[312]. Because of the introduction of semi-monocoque design in aircraft structures and the recent employment of integral stiffener designs in rocket and missile structures, interest has shifted to more economical and efficient stiffened shell structures. Again, when optimization procedures for weight reduction are considered, the stability of the shell structure becomes important. As with unstiffened shell research, the effects of imperfections on the performance of the shell structure became one of the key parameters to study.

3.2.1 Equilibrium Methods

Prior to the 1960's the popular approach in stability analysis of a stiffened shell structure was to examine the response of the shell as two components[93,289,308,309]. The stringer and ring components that are comprised of beam elements were analyzed separately from the skin. It was assumed that the stringers would carry the axial thrust and bending, while the skin would transmit only inplane shear. Perhaps the shear flow method of analysis had something to do with this assumption. Later, in an attempt to more accurately define the response of the two components, the membrane capacity of the shell was included in the response, and the overall response of the shell was examined.

As the number of stiffening elements increases, one popular method that has been used is to alter the properties of the shell by "smearing" the stiffener properties into a combined orthotropic shell. This procedure has some drawbacks because one may lose sight of the "thinness" assumption of the shell inasmuch as the effective thickness would increase, which in turn would tend to increase the effects of transverse shear. Further, when considering the stability of the reinforced shell, the discreteness of the stiffeners may tend to play an important role. With this approach fully explored, the next extension would be the fully coupled problem.

In 1947, van der Neut[310] determined that for longitudinally and circumferentially stiffened cylindrical shells, the eccentricity of the stiffener had primary importance -- inside stiffening reduced the buckling load up to one third of the case without eccentricity. (The greater efficiency of outside stiffening, though apparent from the general formula, was not explicitly mentioned by van der Neut.) This work was done in 1942 and was meant to apply to aircraft fuselages.

Later, Singer, et al.[34,265] presented a similar finding -- that outside stiffeners were more efficient than inside stiffeners, and centrally placed stringers were the least efficient. Nash[219], using small deflection theory, examined ring stiffened shells under external pressure such as used in submarines. Bijlaard[47] accounted for the torsional restraint of the rings under the same loading condition. Becker[44] used the linearized Donnell equations for the stability analysis of stiffened cylinders subject to axial compression, torsion and hydrostatic pressure.

In 1960, Thielemann[297] developed a nonlinear theory for the buckling of orthotropic shells. He examined the nonlinear post-buckling behavior of cylinders and considered an infinitesimal imperfection and radial displacements, and ignored boundary conditions. Through a Galerkin procedure, the classical buckling load was defined for cylinders with various stiffening characteristics. By examining the effect of these stiffening characterizations, Thielemann determined that the resulting deformation patterns, especially those that had short wave lengths in the longitudinal and circumferential directions, thus forming a diamond shape pattern, was strongly influenced by the direction of stiffening.

The key parameter that Thielemann had used in his assessment is γ, which is the ratio of the products of bending and membrane stiffness in each direction. For an isotropic unstiffened shell, $\gamma = 1$. If a cylinder is stiffened by longitudinal members, $\gamma > 1$. For ring stiffened cylinders, $\gamma < 1$. The typical behavior of an axial stiffened and circumferential stiffened shell subjected to an axially compression is depicted in Figures 12 and 13. As seen in a comparison in Figure 12a and Figure 12b, a dramatic reduction in the load parameter λ (normalized to the classical value λ^*) is experienced. Similar reductions are experienced in the end shortening, δ and the classical value $\delta_{c\ell}$. The effects of axial stiffening are similar to that of internal pressure, while for circumferentially stiffened shells, only a slight effect is experienced (Figures 14 and 15).

Thielemann has shown by normalizing with extensional (A_{ij}) and bending (D_{ij}) stiffnesses that the classical buckling load, $N_{c\ell}$, connected with a checkerboard buckling pattern is

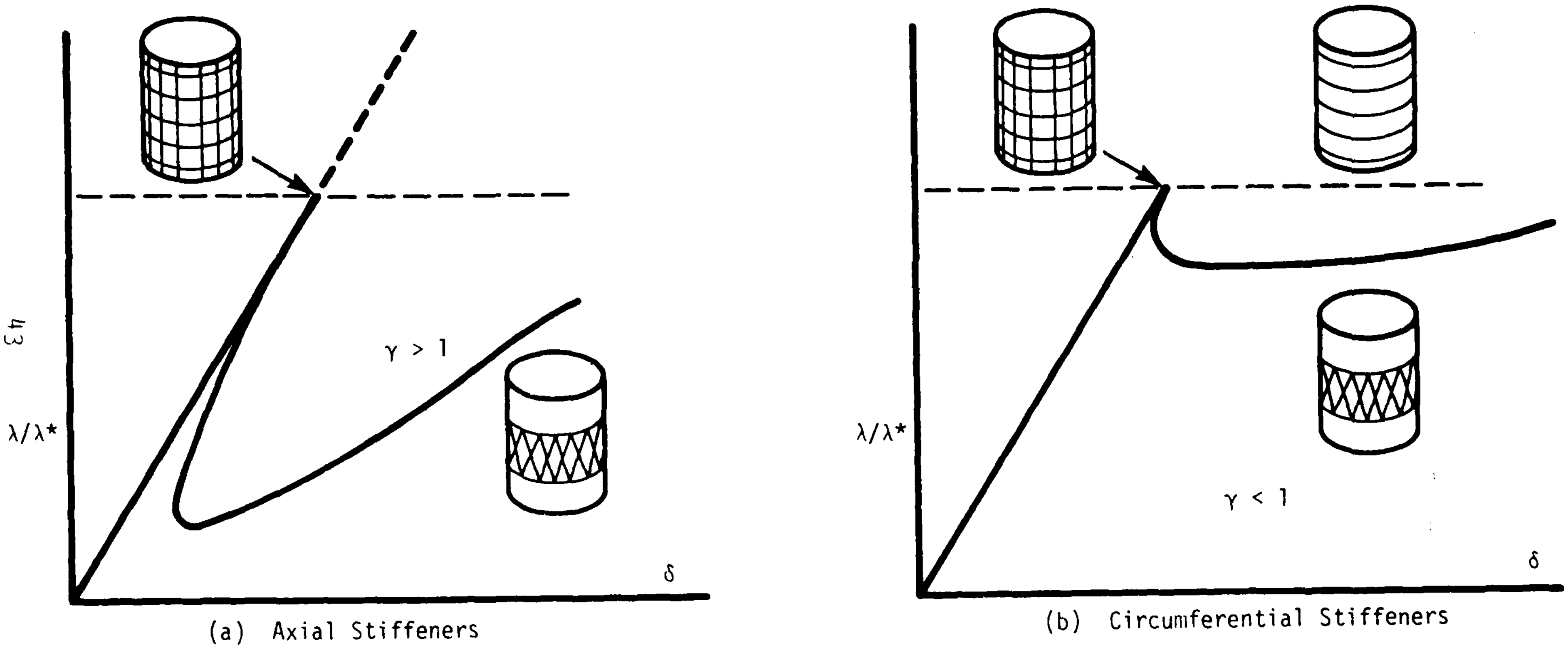


Figure 12. Typical postbuckling behavior of axially loaded orthotropic shell.

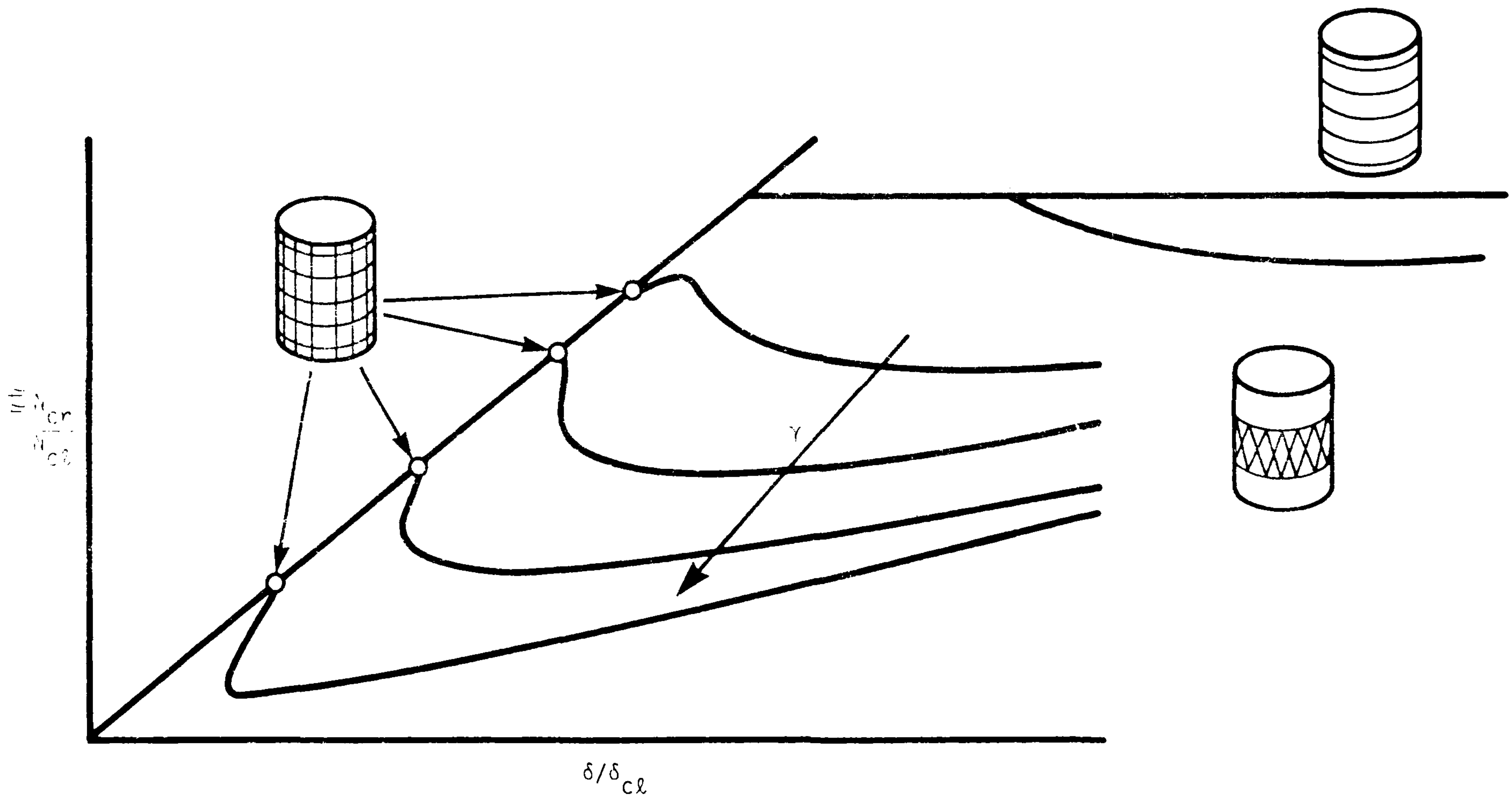


Figure 13. Postbuckling functions of stiffened cylinders axially loaded.

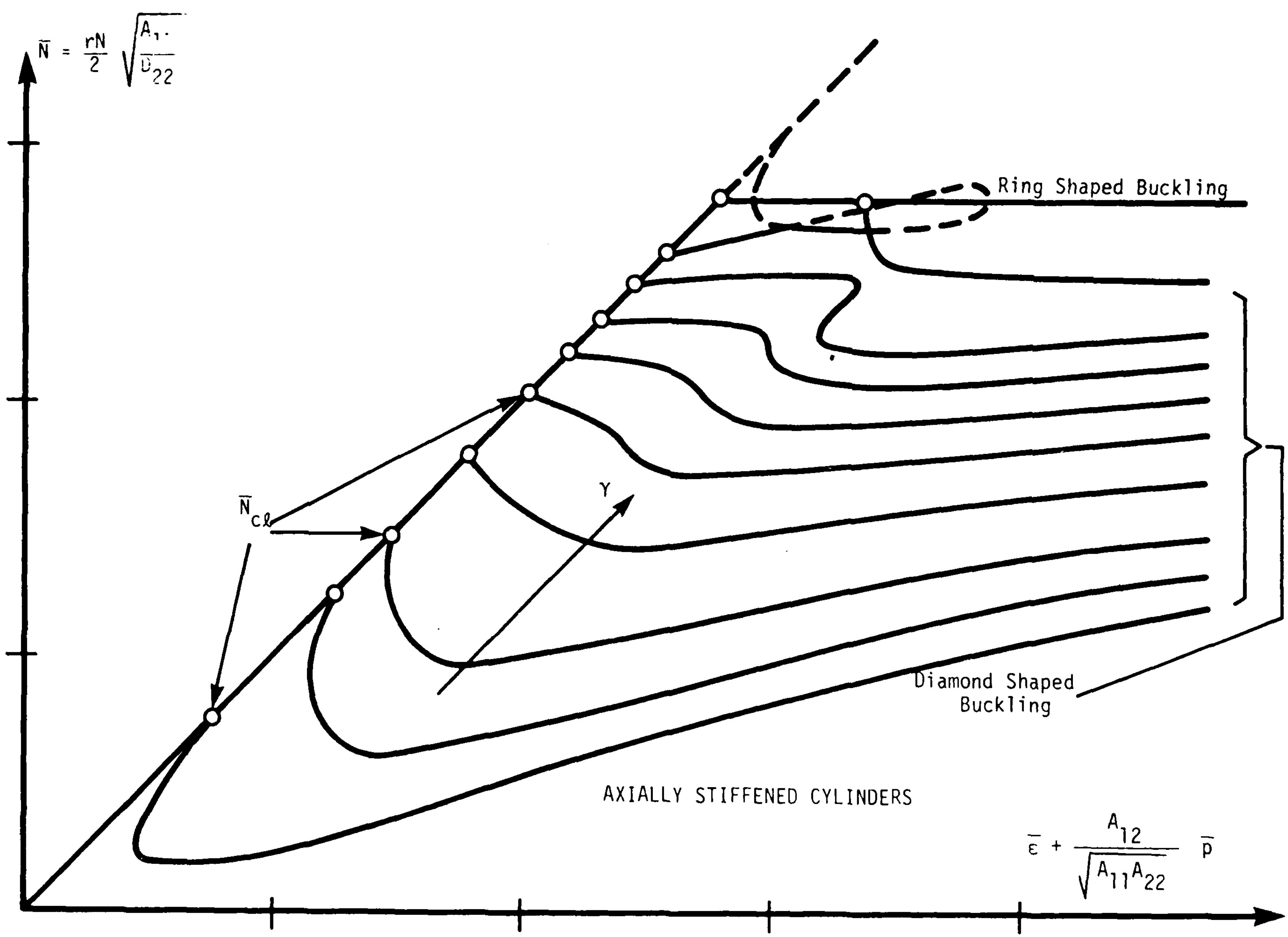


Figure 14. Postbuckling functions of axially compressed pressurized orthotropic cylinders - axially stiffened.

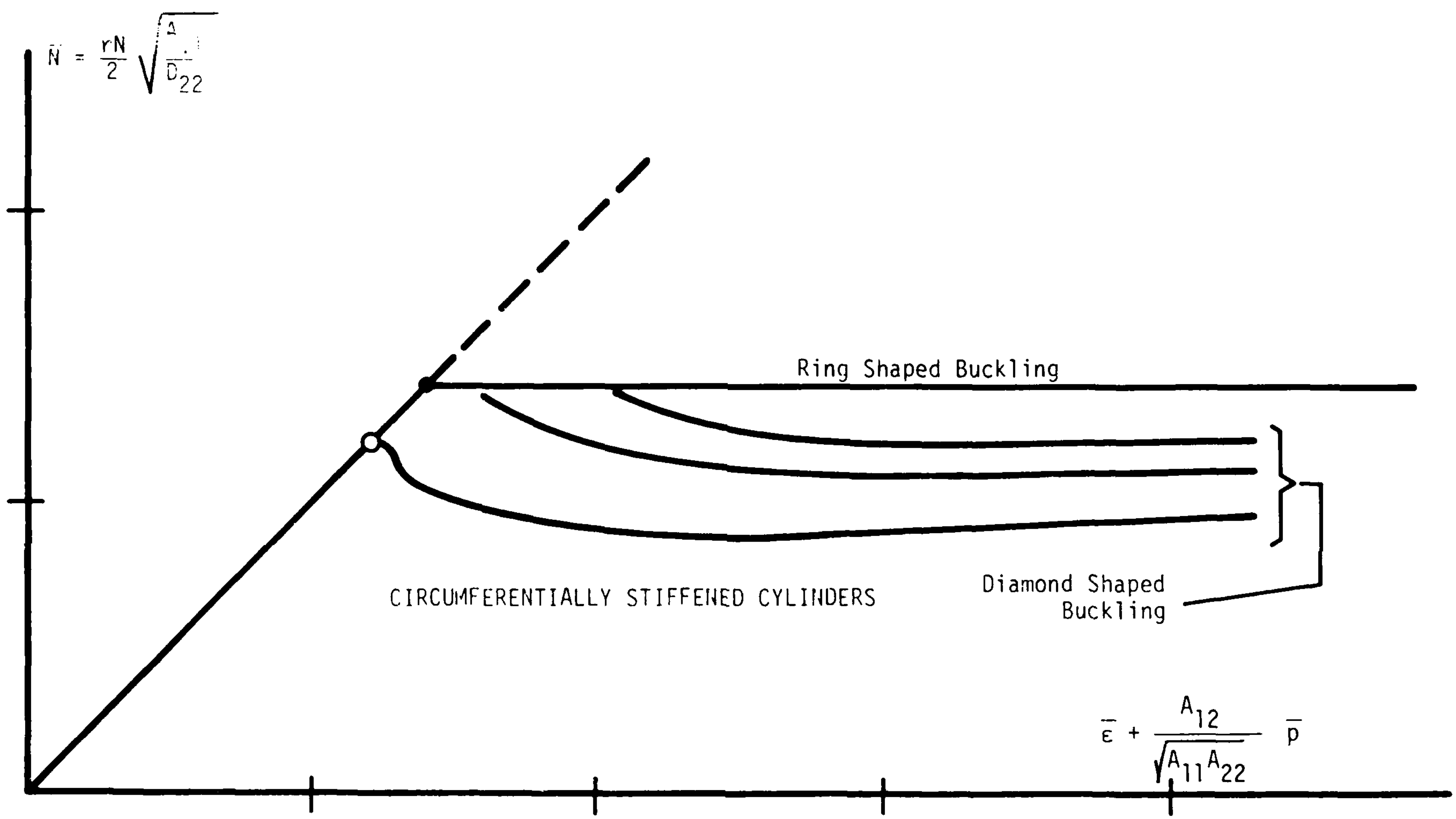


Figure 15. Postbuckling functions of axially compressed pressurized orthotropic cylinders - circumferentially stiffened.

increased appreciably when the internal pressure $\bar{P}$ is increased. The ring buckling load, $N_{c\ell_o}$, is the upper limit. At the points characterizing the classical loads, the postbuckling functions corresponding to diamond-shape buckling bifurcate, and for large values of the end shortening $\bar{\epsilon}$, they asymptotically approach the ring buckling load.

Smith, Spier and Fossum[278] generated a number of parametric studies for an unpressurized stiffened cylindrical shell with clamped boundary conditions based upon Thielemann's method. A similar study was made by Milligan, et al.[208], for axial compression, torsion and hydrostatic pressure. Almroth[5] assumed a five-term displacement function for the radial displacement and re-evaluated Thielemann's method for different extensional and bending rigidities. Later, the influence of stiffener eccentricities were examined through the coupling of the membrane and bending and twisting coefficients of orthotropic shells[286].

Discreteness of ring stiffeners was investigated by van der Neut[310] for the cases of short longitudinal wave length. He found that the error in the buckling load for smeared rings is only about one percent too great when the average half wave length is greater than twice the ring pitch. With smaller wave lengths, the error increases -- a five percent error of a one and one half ring pitch. Block[48-52] later investigated the effects of eccentricity and discreteness of an eccentrically stiffened shell and found that the predicted buckling load may be substantially affected by considerations of prebuckling deformations, eccentric loading, and discreteness of ring stiffeners.

One of the major efforts in understanding the response of stiffened shells has been performed at the Israel Institute by Singer[266-275] and his associates, Baruch[34-37], Haftka[124], Rosen[248-253] and Weller[320-327]. Over thirty reports have been generated on the theoretical and experimental verification of buckling of cylindrical shells. Stringer construction with non-uniform cross-sections were examined. A rigorous analysis that takes into account the discreteness of the stiffener was performed. Unlike Block's findings, Singer shows that unless the number of rings is less than three or four for a shell with L/R > 2, the difference between the "smeared" and "discrete" theories is negligible.

The analysis procedure used by Singer, et al., is to examine the variations of the equations of equilibrium in terms of displacements, (i.e., equilibrium methods). The stress and moment resultants are given in terms of the displacements for the circumferential and longitudinal harmonic.

Through a Galerkin method, a system of determinates results in a tridiagonal symmetric eigenvalue problem. Singer claims that the accuracy of the solution to these problems is quite good, even up to a matrix order of 100; however, the effects of an imperfection is not examined.

3.2.2 Koiter's Energy Methods

Brush[58-59] represents the stability problem of an orthotropically stiffened cylindrical shell as the statement of two adjacent equilibrium states -- one unbuckled, and the other in a buckled state. The solution procedure follows Koiter's approach by stating the potential energy increment in terms of the Euler equations for the second variation of the potential energy. The restriction is that only simply supported shells are considered, although the torsional rigidity of the stiffeners are included. The solution is obtained using a finite difference representation of the energy increment. Stephens[283] examined the imperfection sensitivity of cylindrical panels and accounted for the torsional rigidity of a longitudinal stringer. Again, Koiter's aproach was used.

In a series of studies, Budiansky[62,65], Hutchinson[151,152] and Stephens[283] examined the postbuckling behavior of stiffened cylindrical shells. The effects of eccentricities, prebuckling deformations, imperfections, boundary conditions, pressure and other loading conditions were considered. In all studies presented by this group, the orthotropic properties were based upon a "smeared" representation of the stiffeners.

Hutchinson and Frauenthal[153] developed a postbuckling analysis for orthotropic shells that parallels that presented for a uniform shell represented by Eqs. (12) through (19). The development involved the Donnell equations written in the form:

For compatibility -

$$L_H[F] - L_Q[W] - W^2_{,xy} + W_{,xx}\, W_{,yy} + \hat{W}_{,yy}\, W_{,xx} - 2\hat{W}_{,xy}\, W_{,xy} = 0 \tag{29}$$

For equilibrium -

$$L_D[W] + L_Q[F] - F_{,xx}W_{,yy} - F_{,yy}W_{,xx} + 2F_{,xy}W_{,xy} - F_{,xx}\hat{W}_{,yy} - F_{,yy}\hat{W}_{,xx} + 2F_{,xy}\, \hat{W}_{,xy} + p = 0 \tag{30}$$

where the operators L_H, L_Q, L_D are functions of the shell, stringer and ring properties.

The above equations are in terms of the Airy stress function, F, the radial displacement, W, the imperfection function, $\hat{W}$, internal pressure, p, shell properties (thickness, modulus of elasticity and Poisson's ratio) and stiffener properties. For a given set of boundary conditions and a state of stress, three solutions of the W - F function must be determined.

It is assumed that each function has the form:

$$W = \frac{\nu\sigma R}{E} + w^* + w \qquad F = -\frac{1}{2}\sigma h y^2 + F^* + f \tag{31}$$

The first term in each function corresponds to the membrane state. The starred terms correspond to the prebuckled state ($W^{(o)}$, $F^{(o)}$); and the last term corresponds to the initial postbuckling stage and is comprised of two orthogonal states - $W^{(1)}$, $W^{(2)}$ and $F^{(1)}$, $F^{(2)}$.

Again following Koiter's basic approach, each function can be expanded about the classical load in the form of Eqs. 17 and 18. For a symmetric system, in Eq. 18,

$$a = 0$$

and

$$b = \frac{F^{(2)}*(W^{(1)},W^{(1)}) + 2F^{(1)}*(W^{(1)},W^{(2)})}{P_c[\dot{F}_c^o*(W^{(1)},W^{(1)}) + 2F^{(1)}*(\dot{W}_c^o,W^{(1)})]} \qquad (32a)$$

with the notation:

$$A*(B,C) = \int_s [A_{,xx}B_{,y}C_{,y} + A_{,yy}B_{,x}C_{,x} - A_{,xy}(B_{,x}C_{,y} + B_{,y}C_{,x})]ds \qquad (32b)$$

and

$$(\dot{\ })_c = \frac{\partial(\ )}{\partial P}\bigg|_{p=p_c} \qquad (32c)$$

It follows that an imperfection of amplitude $\bar{\delta}$ will define the load factor just prior to buckling. Koiter has shown that the value of b determines the stability of the shell, per Eq. (20) for cylinders, repeated here as:

$$(1 - \frac{\lambda_s}{\lambda_c})^{3/2} = \frac{3\sqrt{3}}{2}(-\bar{b})^{1/2} \left|\frac{\bar{\delta}}{h}\right| \frac{\lambda_s}{\lambda_c} \qquad (33)$$

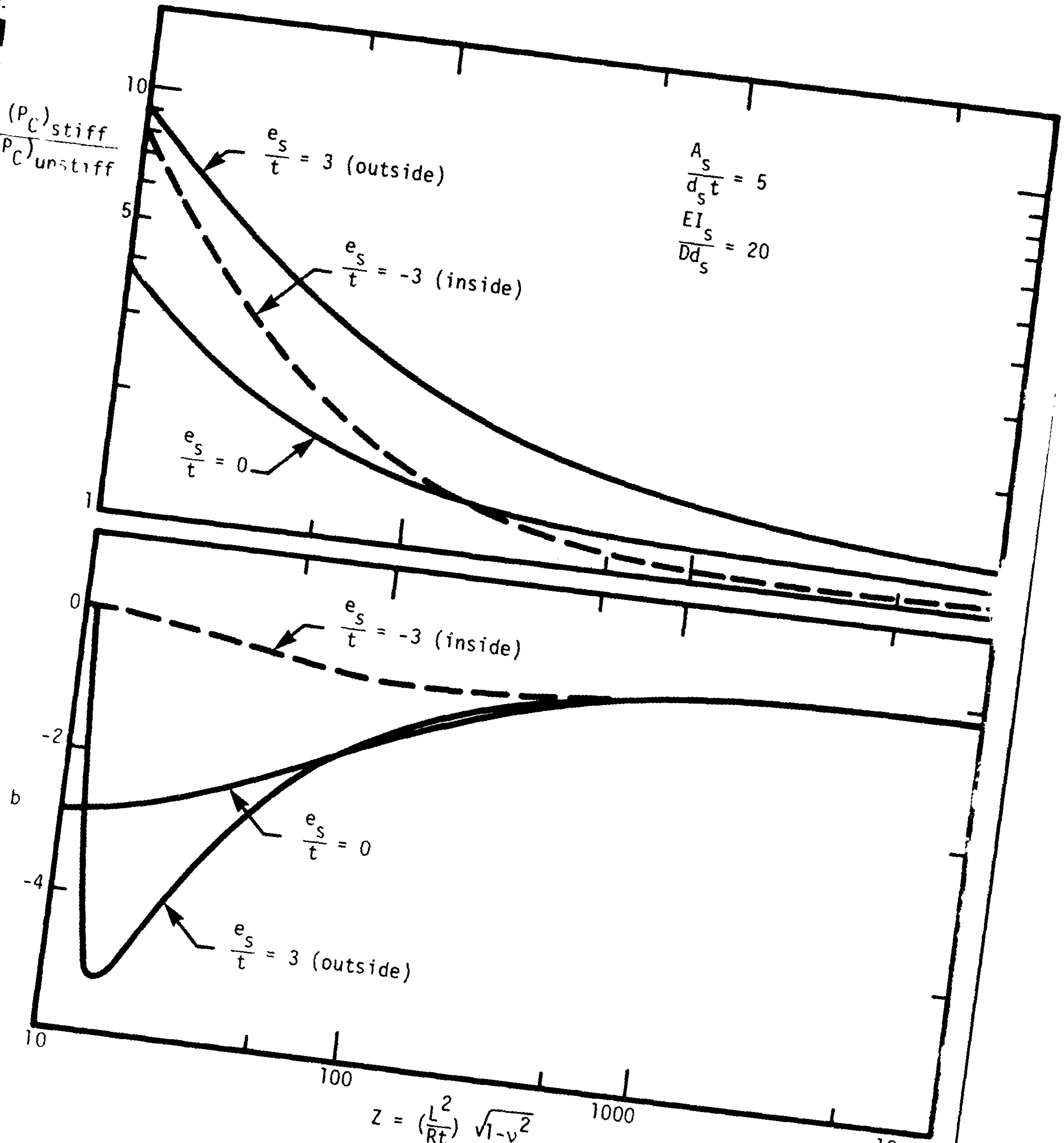
with b replaced by $\bar{b}$.

When the prebuckling state is pure membrane $\bar{b}$ = b, but when prebuckling nonlinear states may be important:

$$\bar{b} = \frac{b\ \{F_c^o *(W^{(1)},W^{(1)}) + F^{(1)}*(W_c^o,W^{(1)})\}^2}{\{P_c^o\ [\dot{F}_c^o*(W^{(1)},W^{(1)}) + 2F^{(1)}*(\dot{W}_c^o,W^{(1)})]\}^2} \qquad (34)$$

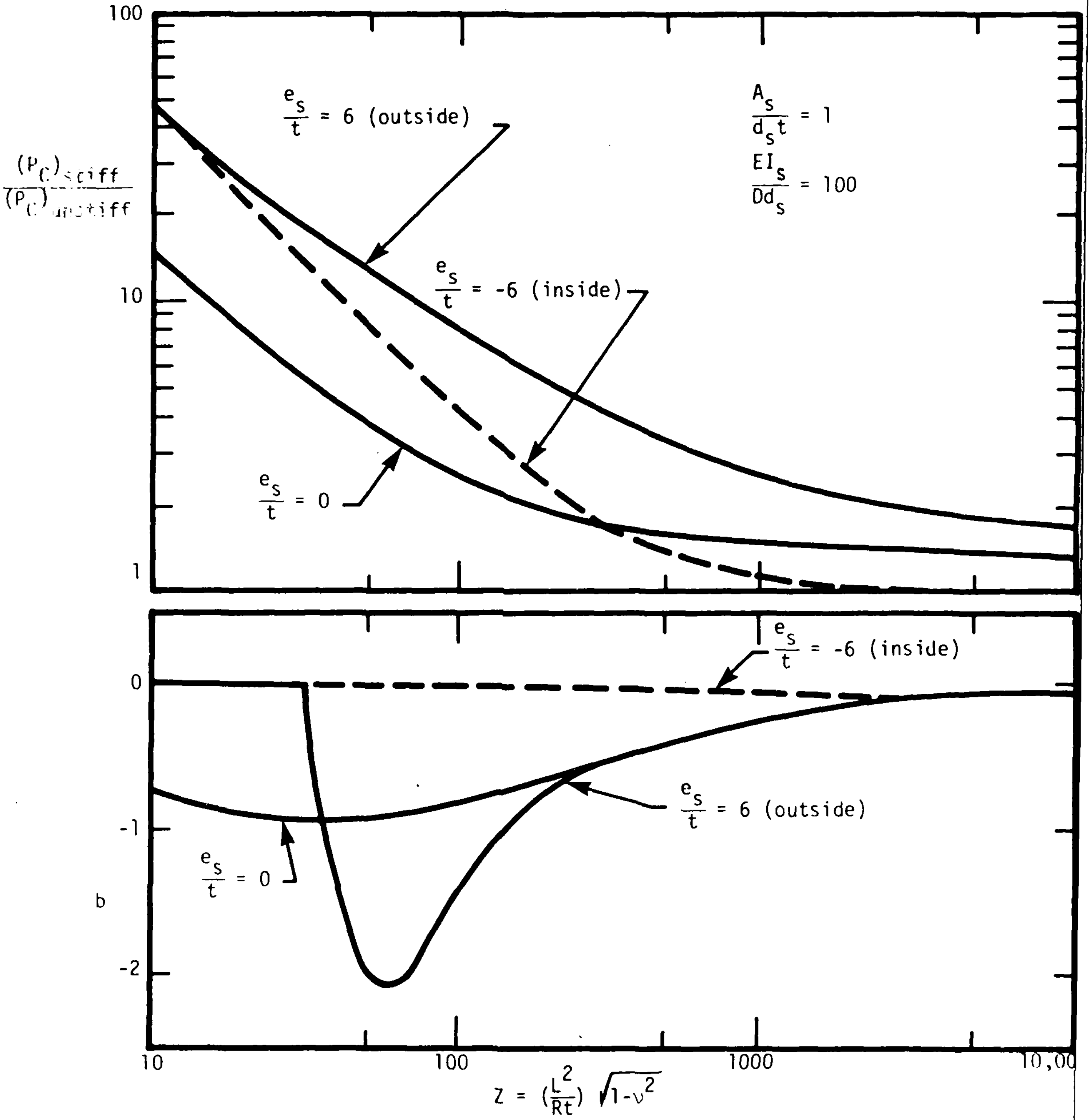
After much algebra[154], the functions W and F were evaluated for a simply supported shell. For the axially stiffened cylinder under axial compression, Hutchinson illustrates comparative stiffening effects. The outside stiffened cylinder denoted by its eccentricity e_s is, in general, more imperfection-sensitive than one with inside stiffeners. The buckling load may be well below the classical value for light and heavy stiffeners (denoted by the extensional ratios of cross-sectional area A_s to panel width d_s and stiffener bending rigidity EI_s to plot flexural rigidity, d_sD); see Figures 16-18. Also, the effect of stringer eccentricity is less prominent under hydrostatic pressure than under axial compression (same as Thielemann's, van der Neut's and Singer's conclusions). The same is true for ring-stiffened shells. Hutchinson also concludes that "if one takes into account the predicted insensitivity of the axially stiffened cylinder in the lower range of Z (Batdorf's parameter) and the sensitivity of the ring-stiffened specimens, then the advantage of axial stiffening is even more pronounced."

Several studies have been made to examine the imperfection sensitivity of stiffened shells when overall and local buckling modes couple. Byskov and Hutchinson[71] suggest an approach extending Koiter's method. Koiter[176] demonstrated that Byskov and Hutchinson, in a preliminary report, ignored a crucial term in the mode interaction which had an important effect on the result. This criticism brings out one of the major disadvantages of the asymptotic methods -- namely, the importance of the terms omitted in the expansion cannot always be measured unless the system is resolved with the greater complexity included. Koiter's method calculates the slope and curvature of the post-buckling path at, or immediately adjacent to, the bifurcation point. In some cases, the asymptotic representation of the post-buckling path, based upon these two quantities, may be misleading; hence, imperfection sensitivity results might be asymptotically valid but practically misleading.



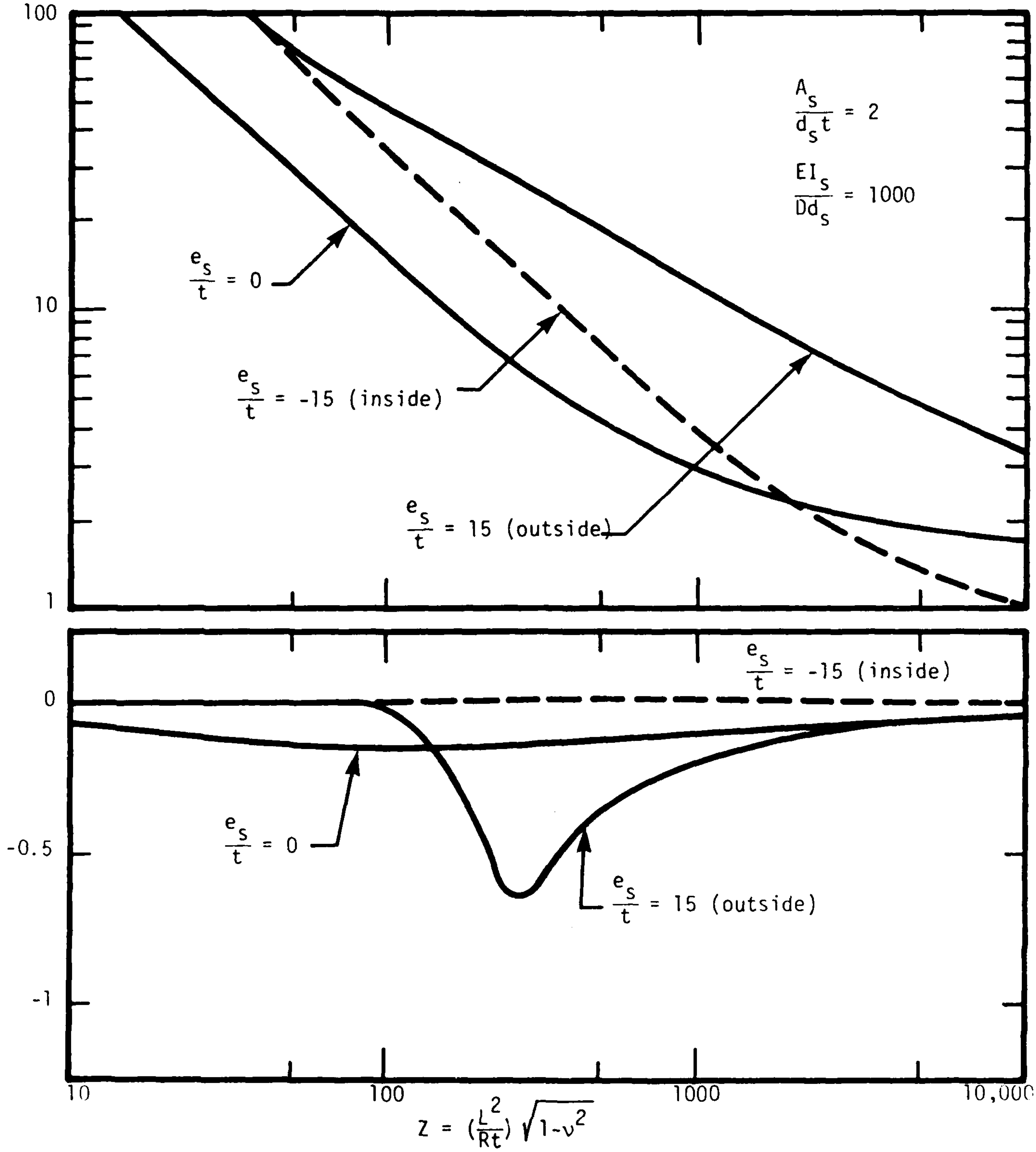
(a) Light Stiffening

Figure 16. Classical buckling and imperfection-sensitivity of simply supported, axially stiffened cylinders under axial compression.



(b) Medium Stiffening

Figure 16. Continued.



(c) Heavy Stiffening

Figure 16. Continued.

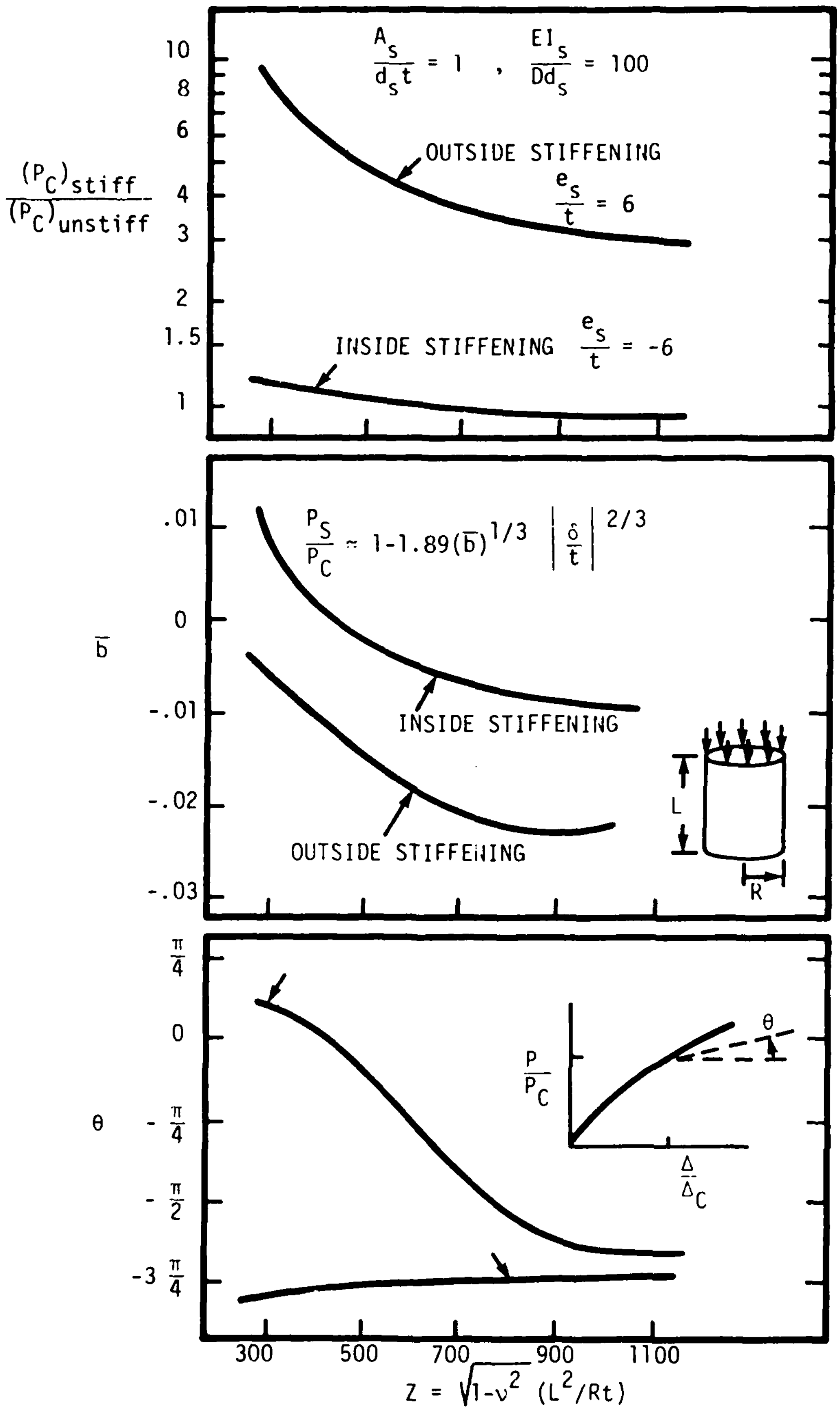


Figure 17. The effect of stringer eccentricity on the buckling and postbuckling behavior of axially stiffened cylindrical shells which are simply supported at the skin middle surface and loaded in axial compression

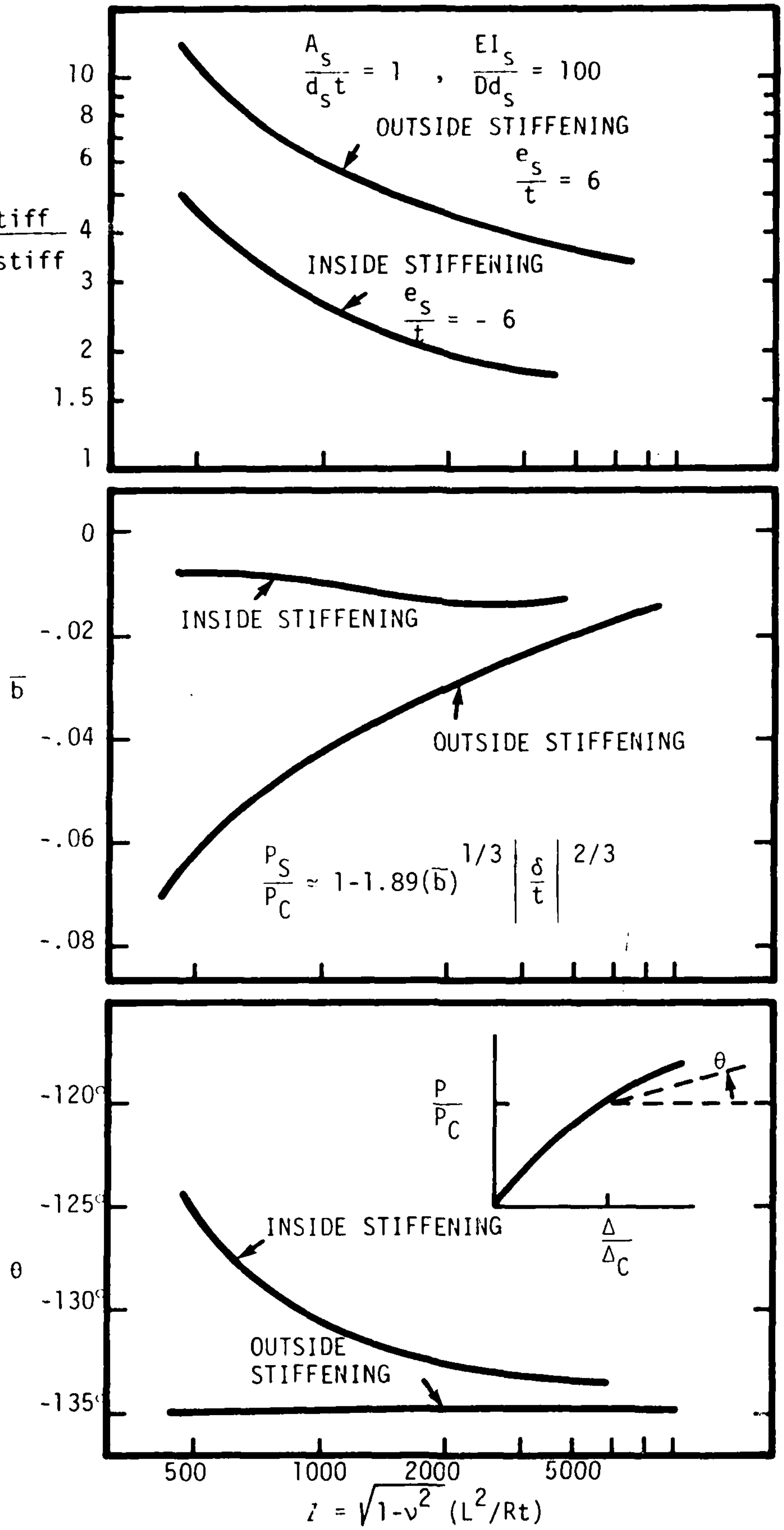


Figure 18. The effect of stringer eccentricity on the buckling and postbuckling behavior of clamped, axially stiffened cylindrical shells which are loaded in axial compression.

The imperfection sensitivity of stiffened shells is further complicated by the fact that multi-mode interaction tends to mask the effects of the individual components on individual or single mode buckling behavior. Tvergaard[309] reviews this area and suggests that single mode asymptotic techniques are inaccurate for imperfection sensitivity predictions. van der Neut[311] shows that for built up columns where the separate local and overall buckling occurs at the same stress, the interaction of the two modes causes unstable equilibrium with the effect that local imperfections yield a reduction of the overall buckling load. This result was confirmed by Koiter and Kuiken[179]. Koiter and Pignataro[178] examined stiffened panels and concluded that only a small imperfection sensitivity should be realized. Koiter and van der Neut[181] show that the earlier analysis[178] is inadequate for panels where in the local mode stiffener wall deflections and skin deflections have an equal order in magnitude. The first two local modes, together with the overall mode, are needed for assessing the effects of the interaction.

4.0 NUMERICAL METHODS

There have been some attempts to evaluate the imperfection sensitivity of shells numerically, particularly for shells of revolution. Cohen[83-86] used Koiter's unique mode bifurcation buckling theory for the axisymmetric nonlinear prestressed states of a ring-stiffened shell of revolution with an arbitrary imperfection shape. Haftka, et al.[122,123], used the modified structure method for two-dimensional finite elements and found some limited success in this approach. They found that the nonlinear states of stress prior to buckling caused buckling to occur at a load significantly different from that obtained through linear stability predictions. Maewal and Nachbar[199] examined the post-buckling behavior of a clamped uniform cylindrical shell under uniform axial compression using the same method. They required a high order isoparametric finite element to gain the accuracy required for imperfection sensitivity predictions.

The existence of closely spaced bifurcation points leads to a complex and sometimes time-consuming search for the minimum eigenvalue. Again, the theories of Thompson seem to apply. If the analysis procedure would permit response in all Fourier harmonics, then a minimum imperfection-sensitive calculation could be accomplished. If discrete response Fourier harmonics or limited models are used, then the minimum response harmonics and all important coupled modes must be known _a priori_.

The use of the direct numerical evaluation of the nonlinear behavior of shells has been attempted with some limited success. Several computer programs exist that are capable of examining the static imperfection sensitivity of shells -- BOSOR[66], STAGS[7], FASOR[87], and NBALL(SATANS)[80]. The last three permit general imperfections, while BOSOR permits only axisymmetric imperfection. Only STAGS and NBALL permit general loading to be considered. STAGS can examine the general two-dimensional shell form, while NBALL is limited to shells of revolution.

There have been some special programs that have been written in an attempt to reduce the computer run time for a geometric nonlinear analysis. Bauld and Satyamurthy[43] have written a program based upon the finite difference energy method (similar to that used in STAGS) for a cylindrical panel. The examination of the sign of the final determinant is used as a key for the limit (bifurcation) point. Since a Newton-Raphson procedure is used, only the static nonlinear prebuckled states up to buckling can be studied. To examine beyond prebuckled states, the dynamic option must be used. Only STAGS and NBALL have this capability. Kicker[165] has written a computer program that examines the linear bifurcation predictions for a cylinder with longitudinal and circumferential stiffeners or panels with edge stiffeners using a selected theory of Flügge, Timoshenko or Donnell. Smeared stiffener properties are assumed, and eight possible boundary conditions are available.

An interesting comparison between Koiter's initial postbuckling theory and numerical results from the nonlinear analysis program NBALL has been made by Stillwell and Ball[284]. From Fitch and Budiansky[106] the initial slope of the bifurcation branch of the equilibrium path on a load-displacement plot had been predicted for a uniformly loaded spherical cap with a rise height parameter of $\lambda=8$. The numerical results were obtained by assuming a slight eccentricity of load of the form:

$$p = p_o (1 + \varepsilon \cos n \theta)$$

By first obtaining the minimum critical Fourier harmonic, n, Stillwell and Ball determined the maximum response for the shell under different increasing eccentricities to coincide with Fitch's predictions. Since the eccentricities were small, two important conclusions can be drawn:

1) Numerical methods can be used to accurately predict the bifurcation loads of shells of revolution when imperfection sensitivity is of concern.
2) Koiter's method is applicable for small imperfections.

5.0 EXPERIMENTS

The ultimate test of a particular theory or numerical method is the comparison of predictions to experimental data. Without the full understanding of the static stability behavior of shells, experiments were performed with limited success. From the early tests of Robertson[245], Donnell[97], Tokugawa[305], Lundquist[196], and others such as Bridget[54], Holmquist[139], Ikeda[155], Kanemitsu[160], and at NACA[210,226,282] in the 1930's, and by NACA-sponsored tests in the 1940's[12-18,77,92,93,101,102,132,140,141] inconsistent data was reported. Since the cylinder possesses the greatest sensitivity to axial load, this shape was used for most of the experimental studies in the 50's and 60's at NASA[23,69,74,100,209,231,234,244,285] and by Galletly[110,111], Allen[2], Becker[46], Corum[89], Harris[128], Kirstein[166], Singer[263], and Wenk[328]. The David Taylor Model Basin performed several tests on controlled spherical specimens[183,222,256] as well as cylinders[129,238]. Other shapes such as spheres[1,193], ellipsoids[19,33], toroids[158], and cones[305-308] by Holownia[142], Singer[267], Williams[329] and Weller[321] have been studied and test data has been compiled. The influence of stiffeners has also been studied experimentally. The influence of the shape and attachment was considered by Baruch[36], Dow[99], Card[72,73], Corum[88], Medgley[204], Singer[250,276] and Weller[320-327].

Because the factors that influence buckling of shells have not been completely understood, experiments have been performed and data collected that presents conflicting results. The collection of data for the ratio of buckling pressure to classical ($P/P_{c\ell}$) for the clamped spherical cap[161] having various rise heights, λ. Figure 19 is an example of the scatter that can be anticipated. The size of the specimen is usually restricted by the load-carrying capacity (more importantly, the stiffness) of the supporting test fixtures. To reduce this influence, nonmetallic test specimens have been used. PVC, acrylics, rubber, mylar, and other plastic materials have been candidates. For thin metal shell structures, the electroforming process has also been used[76,258-260].

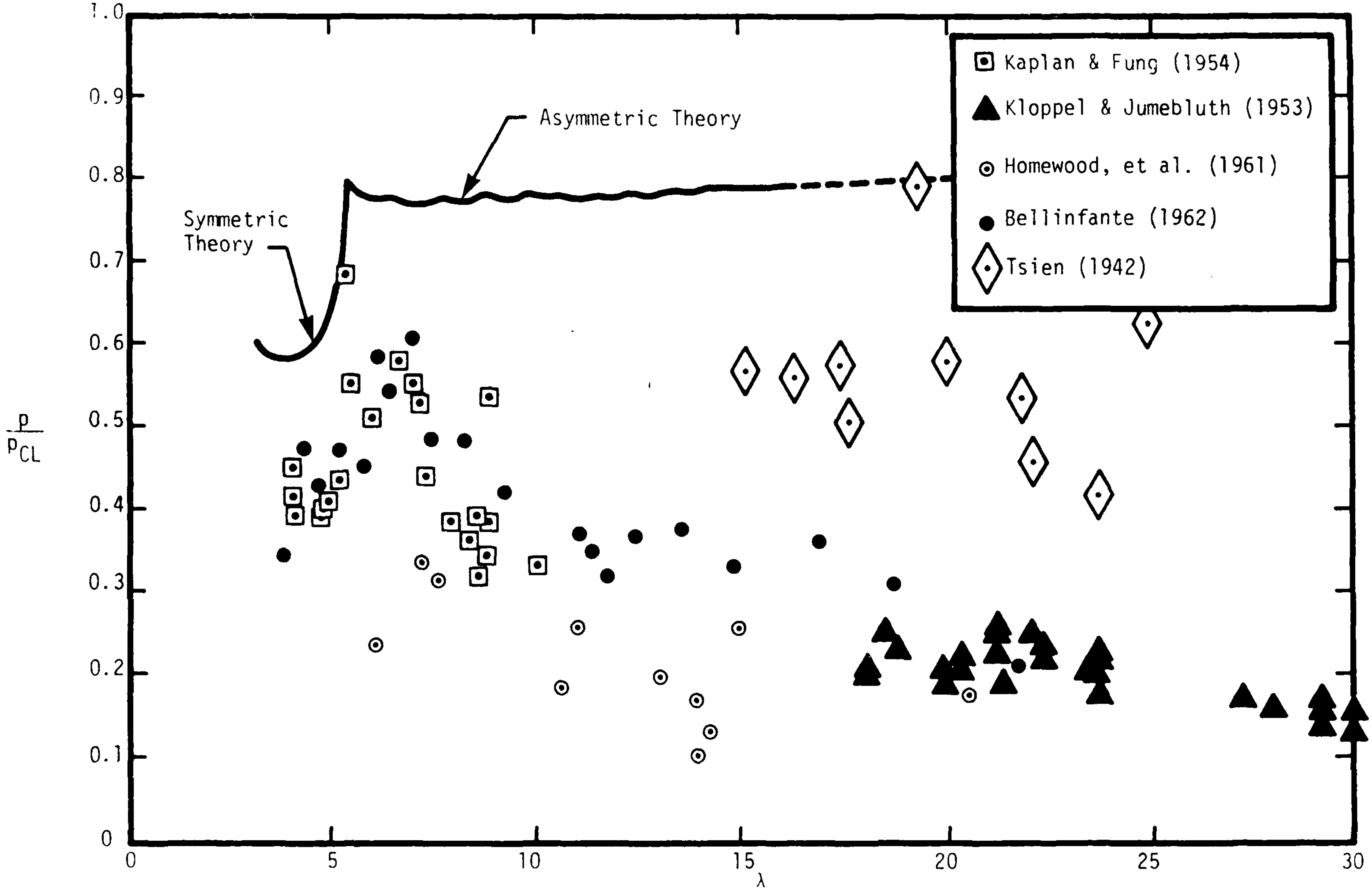


Figure 19. Early experimental results for clamped shells.

The unfortunate thing is that the subscale tests are also subject to the same physical laws as their prototypes. This means that the same seven control parameters suggested by Thompson apply to the experimental specimens as well. Further, the scaled tests may amplify one imperfection parameter that is not actually contributing in the full-scale specimens. Since shape imperfections are generally used as one of the controlling parameters, special care should be taken to insure that only shape imperfection is being introduced. The control of these parametric deviations becomes more critical. The main unknown factor in specimen evaluation is the determination of the true state of stress. Small deviations in material thickness or residual stresses may mask the desired results. Further, when plastic materials are employed, the uniformity of material "elastic" coefficients may be difficult to achieve or accurately assess.

One of the major problems facing the experimentalist investigating the buckling of shells is that of fabricating the "perfect" test specimen. The fact that there are some shell/load combinations (especially for large R/t ratios) that are very imperfection sensitive is borne out by the studies of Carlson, Sendelbeck and Hoff[76] from which Figure 20 is extracted. One can observe the significant scatter obtained for different specimens and the authors' achievement of better correlation between experimental and theoretical values of shell buckling pressure as their fabrication processes improved for the later specimens. They used an electroforming process to manufacture their test specimens, which were essentially complete spherical shells.

For large radius to thickness ratios ($R/t > 1000$) and near perfect complete spherical shells, Carlson, Sendelbeck and Hoff observed that most of the shell surface becomes unstable at the same pressure; and dimples, covering much of the surface, develop simultaneously. However, for more severe flaws, single dimples form and appear at the individual flaw locations. It has been only in the last few years that investigators have tried to

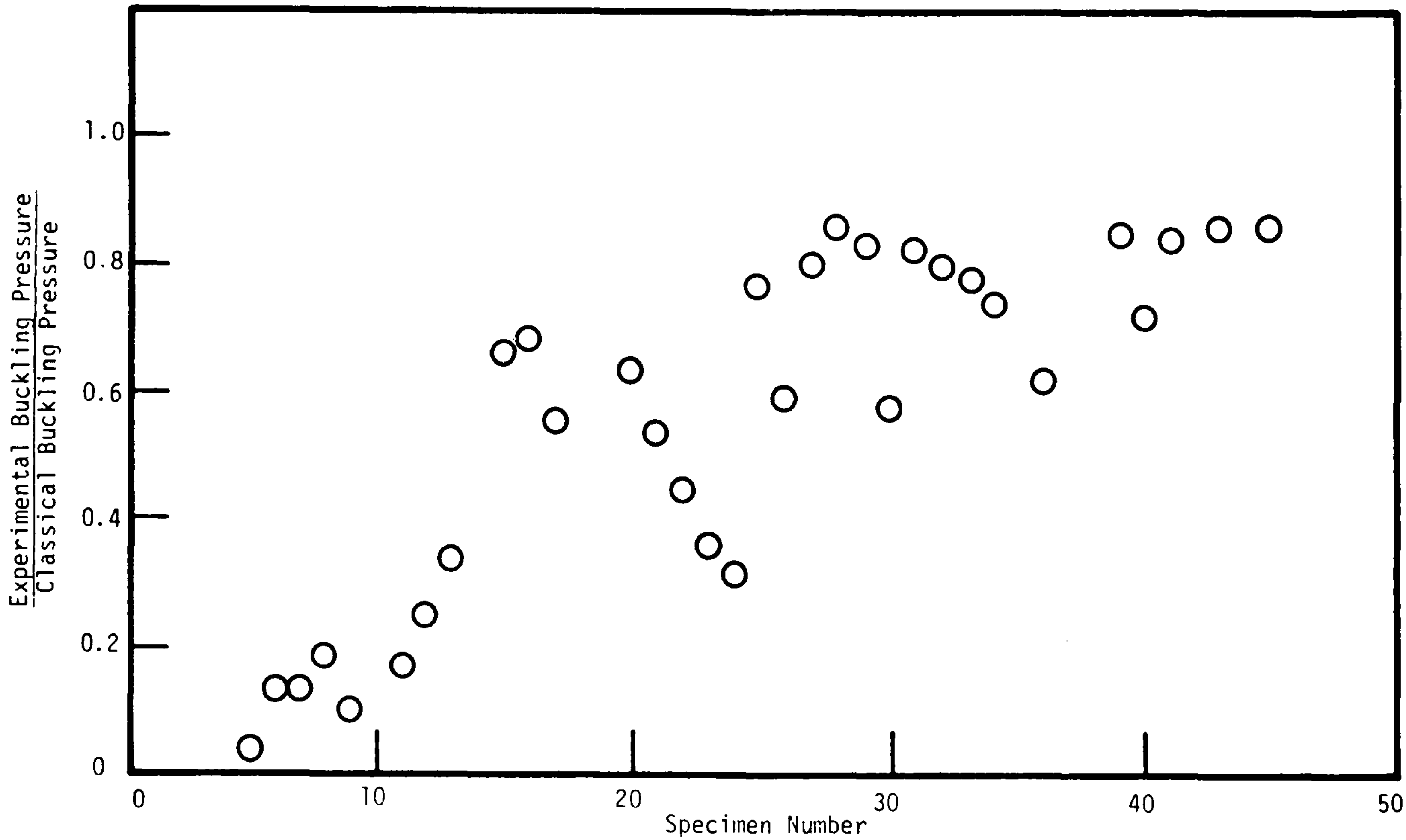


Figure 20. Summary of sphere test performance.

ascertain the nature of imperfections in fabricated shell structures, either full-scale or model-sized (Arbocz[24-26], Horton[144-146] and others[184,228]). In 1969, Arbocz and Babcock[24] reported on the results of buckling experiments in which, for the first time, the actual initial imperfections and pre-buckling growth of electroplated, isotropic shells were measured and recorded by means of an automated scanning system. In 1971, Singer, et al., published the results of similar imperfection surveys on ring and stringer-stiffened shells. Parmerter[228] in 1972 published results on an interesting holographic method of recording shell deformations for shallow sperical caps. The work of Krenzke and Kiernan[184] published in 1965 addresses the problem of imperfections in partial nonshallow spherical shells with imperfections machined into the specimen.

With this experience, Horton[144,146] examined the collapse of an axially loaded cylinder. It was suggested that as the load increased, the number of buckles increased with the change following a normal Gaussian distribution. Thus, Horton concluded that a random variation in the imperfections had to have existed. If Thompson's supposition were considered, it would suggest that the number of equilibrium paths are so close that the randomness would be attributed to the number of possible equilibrium paths. Singer has shown that random type imperfections can be correlated to linear buckling theory. Arbocz has demonstrated that with care in the description of the imperfection, nonlinear theory can be applied to give very accurate results. Therefore, we have contradicting statements for the same phenomena.

Babcock[30] gives a rather comprehensive assessment of the experimental methods employed from 1950 to 1972. He makes a clarion call for experimenters to carefully design the test and to record all detailed information on the shell specimen and its loading environment. The call appears to have gone unheeded. Because the archive publication requirements have restricted the length of many articles, the necessary detailed information on specimen configuration, etc., must usually be omitted from the article. By the time an inquiring analyst pursues the detailed material, the data has been mislaid, or the data is part of a larger set of data which is under a corporate proprietary classification and cannot be distributed, or the researchers have left the institution.

Fortunately, there are exceptions. Arbocz has continued his studies of imperfection in cylindrical shells and has presented data in a variety of publications while pursuing imperfection correlation studies. Singer and Tennyson and their associates also have contributed to the traceable effects of experimentally determined imperfections and boundary conditions.

With more control on the specimen preparation, it is possible to examine more closely the effects of imperfection. To this end, one method has been developed to prepare specimens with a known imperfection. Boros[53] examined the deteriorating effect of prescribed axisymmetric imperfection shapes on the buckling load of ring and stringer stiffened cylindrical shells under a variety of loading conditions. He developed a process that could introduce prescribed axisymmetric imperfections into stringer stiffened cylindrical shell specimens and maintain responses in the elastic range. Thus, repeatability of results could be demonstrated. The results tend to show that Hutchinson's nonlinear theory is quite applicable for a variety of shell geometries and stiffness ranges of external or internal stiffeners.

In a recent survey on buckling experiments of shells, essentially conducted after Babcock's review, Singer[277] reviewed the relationship between the need for experiment and the use of the modern digital computer. He suggested eight reasons why experiments of the buckling shells should be conducted:

(1) Better understanding of buckling behavior and the primary factors affecting it.
(2) Finding new phenomena.
(3) Obtaining better inputs for computations.
(4) Obtaining correlation factors between analysis and test, and for material effects.

(5) Building confidence in multi-purpose computer programs.

(6) Testing novel ideas of shell construction or very complicated shell elements.

(7) Understanding buckling under dynamic loading and in fluid/structure interaction problems.

(8) Certifying tests of full scale shell structures.

With this survey of data developed in the 70's, Singer outlined eleven areas for further research, including many of the areas covered in the present survey: composites, load interaction, imperfection data banks, mode interaction, boundary conditions, and testing techniques. It is recognized that although adequate computational methods are available for the theory upon which they are based, a need for the verification of the theory itself remains a requirement. This can only be accomplished through verification by experiment.

6.0 CONCLUSIONS AND SUMMARY

It appears that the state-of-the-art of the analysis of imperfection sensitivity of shells is not at a sufficient level to provide an all-encompassing theory that engineers can use with absolute confidence. The theory of Koiter has been generally accepted as one measure in determining the stability characteristics of shells but this theory has proven to be limited. Extensions of this theory by Thompson, et al., have shown some promise to extend the range of shell properties over which their method could be applied, but more importantly they make it possible to identify buckling phenomena that could be overlooked by others.

As pointed out in this survey, the theorectical development on the understanding of imperfection sensitivity of shells can become quite cumbersome even when examining the behavior of simple shapes. The developments of the Koiter postbuckling procedure falls into this category. The conditions set forth by Thompson is almost of the same complexity. However, there seems to be some benefit in developing an understanding for both methods. If semi-empirical methods are to be used, there is always a question as to the condition of the shell under investigation as compared to the accepted test data. Further, one must accept the lower bound philosophy inherent with semi-empirical methods.

As an aid to the reader, a partial categorized cross-reference is provided in Table 1 for the various articles highlighted in this survey. Their order of appearance by no means implies a rating of the material. The categories are subdivided as general theory of shells, state of art review, stability prediction methods, numerical methods used in shell stability analysis and experimental results of different shell geometric.

At present, only one computerized procedure (written by Cohen) permits the study of imperfection sensitivity of an arbitrary shell of revolution under specific loads using Koiter's theory. In order to perform a simple investigation, Cohen's

TABLE 1. CROSS-REFERENCES BY CATEGORY

GENERAL THEORY

Linear Methods

31, 32, 70, 108, 120, 131
168, 182, 192, 194, 198,
240, 281, 303, 304, 313,
317

Non Linear Methods

78, 97, 107, 191, 203,
217, 241, 242, 254, 314
316, 332, 336

Review Articles

61, 67, 68, 79, 109
189, 216, 220, 221
224

Stability Prediction Methods

Linear

Analysis

6, 38-42, 44, 45
47, 93, 134, 135
137, 215, 225, 261
262, 278, 289, 334
335

Semi-Empirical

33, 34, 44-46, 60
81, 112-117, 186
204-207, 277, 328

Non Linear

8, 9, 10, 22, 25, 28, 55
56, 61-65, 71, 82, 105
118, 130, 147-154, 156
163, 164, 169, 170-179
243, 246, 257, 271, 283
287, 288, 291-297, 299-302
307, 309, 311, 312

TABLE 1. (continued)

NUMERICAL METHODS

Finite Element	Finite Difference	Numerical Integration
122-123, 199	7, 43, 66, 80 165	83-86, 87

Experimental

Cylinder	Cone	Spherical	Others
2, 4, 12-18, 23 30, 36, 46, 53 54, 69, 72-75 88, 90, 100, 128 129, 132, 139 140, 162, 196 204, 213-215 231-236, 238 244, 249-253 259, 282, 290 305, 320-327 328	130, 142 145, 260 263, 267 329	76, 159, 167 183, 184, 186 193, 202, 222 256	19, 33, 77 111, 141, 158 186, 296

procedure requires the use of three separate computer programs. This approach has not been widely used because of its complexity and availability. Other computer programs that can compute the buckling load of imperfect shells are available, but the cost of the analysis may become prohibitive.

It is comforting to note that when these numerical procedures are applied to specific controlled problems, limited correlation between the previously mentioned theories and the more general computerized procedures have been made. If the analyst is willing to accept the results from these procedures, he must also be aware of their shortcomings. The stability characteristics of the structure are being evaluated, not the numerical procedure. Further, since most computer procedures have certain computational capacity limitations, the principal components that will develop critical participating responses must be approximately known in advance. Thus, non-critical components need not be considered in the formulation of the problem but there is always a chance of not including a principal component.

The need for a simple computer program for the simple evaluation of imperfection sensitivity is required. This can be done for uniform shells, such as cylinders, but not for general shells or loads. Further, the study of stiffened shells with imperfections, which can be performed on a simplified basis, needs to be made available to the engineer. With these tools, the engineer can at least make some preliminary evaluations without the need of complex relationships or large computer budgets being considered.

More general or unified theories are required to handle the specific case studies required in the analysis of near optimum shell structures. This can only be accomplished through the extension of the accepted theories and the accompanying numerical methods. Again, as pointed out by Singer, improvement in the accuracy of these prediction techniques is required, and they must be supplemented and verified through adequate experiments.

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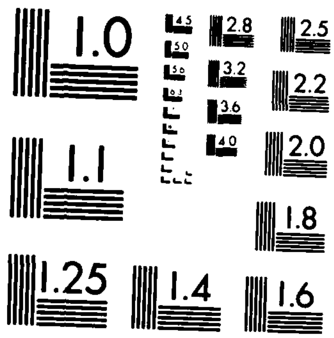

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DISCUSSION

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The review is a thorough and comprehensive survey of the field, attempting to bridge the gap between basic shell theories and practical elastic stability analysis for stiffened shells. This difficult task is accomplished successfully in most of the review, but the following comments are warranted:

1. The possible practical applications of catastrophe theory are not obvious from the brief discussion presented, and the implications of the two theorems require further clarification.

2. The effect of boundary conditions on unstiffened cylindrical shells is briefly mentioned in the review. For stiffened shells, however, the influence of the boundary conditions is of prime importance and may sometimes even overshadow the influence of geometrical imperfections (see, for example, Refs. 273,276, or Refs. A1-A3*), and hence justify some elaboration. The influence of load eccentricity may also be of prime importance (see Refs. 51 or 324).

3. The predominant effect of the boundary conditions on the buckling of stiffened shells, and their similar influence on the vibrations of these shells, motivated extensive correlation studies which yielded a nondestructive tool for definition of the boundary conditions -- the vibration correlation technique. This technique, which consists essentially of an experimental determination of the lower natural frequencies of a loaded shell and evaluation of equivalent elastic restraints that represent the actual boundary conditions, has been developed in recent years and applied to different shells and loading conditions (see Refs. A4-A8).

*The prefix "A" refers to Additional References following Author's Closure.

4. The imperfection-sensitivity predictions shown in Figs. 16-18 of the review are found in later studies (see Ref. 153) to depend strongly on the prebuckling deformations, which can appreciably change the imperfection-sensitivity and the Z (Batdorf shell geometry parameter) at which it occurs (see Fig. 11 in Ref. 268). Furthermore, tests do not verify the predictions of Figs. 16-18 (see Fig. 14 of Ref. A3).

5. Extensive imperfection measurements have recently been carried out on stiffened shells and an international data bank of imperfection measurements has been established (see Refs. 26,28 and A9) which will eventually permit meaningful correlation studies between manufacturing techniques and significant imperfection shapes.

<u>A. van der Neut, Delft University of Technology</u>

The survey deals with an enormous amount of literature and the author has done a very good job. My response on the parts with which the writer is familiar are noted.

The usual approach of establishing the minimal equilibrium load at large deformations was known before the war, due to von Karman and Tsien's paper of 1939 (see Reference 314). Therefore, Koiter deliberately rejected this approach when he chose to focus on the circumstances which trigger collapse and on the load at which collapse occurs. The difference between the two philosophies can be illustrated by the investigation of a car accident where the car crashed into a tree. The parallel to establishing the far post-buckling equilibrium is then the assessment of the amount of damage done by the tree, whereas the real problem is to know what caused the car to deviate from its straight course. During two decades and even during the sixties, when Koiter's theory had received publicity, a vast amount of energy was wasted in studying the final "car damage" before it was generally recognized that the real problem is the initial post-buckling behaviour.

The two-flange-model the writer used in Ref. 310 was particularly apt to call attention to the unfavourable effect of interaction between overall and local buckling, because it exaggerates this effect due to the presence of two independent local buckling modes at the same stress. A later report (see Reference A10) suffers from the same defect.

An alternative approach to the interaction between local and overall buckling in stiffened panels, given by W.T. Koiter and M. Tignataro, avoids the coincidence of two local modes and yields a milder effect of interaction. In later work the writer removed the defect of coinciding local modes by introducing elastic coupling between the two flanges. It lessened the effect of interaction and confirmed Koiter's "alternative approach" for the case where the stiffener walls in local buckling have much smaller deflections than the skin plate. It appeared however when applying the "alternative approach" and the revised model to a panel where the deflections of the stiffener top (e.g., Z-stiffened panel) are of the same order of magnitude as those of the skin, the Koiter method underestimates the effect of interaction. Thereupon, Koiter has improved his analysis by taking into account the first two local modes instead of only the first one. The revised model parameters were adjusted to the results obtained by Koiter for the perfect structure. This model was then used to establish the effect of local imperfections. Next, for the square tube the stability at the bifurcation point against small finite deflections of the column axis and the effect of columns axis imperfections were established. These most recent developments have been the subject of a joint paper by Koiter and the writer and were presented at the "Int. Conf. on Thin-Walled Structures"[181].

AUTHOR'S CLOSURE

I wish to thank Profs. Singer and van der Neut for their added discussions on the survey. Since a great deal of information had to be omitted from this survey, some of the key points that describe the basic assumption behind the theories may not be sufficiently highlighted. To the reader, I apologize.

Singer's first remark with respect to the use of catastrophe theory should be addressed. This very point has raised a few negative remarks by other researchers.

Thompson and Hunt in their book present a rather detailed discussion of how energy methods (not catastrophe theory) can be employed in the evaluation of elastic stability. The two stated theorems permit elastic stability prediction to be made <u>solely</u> from equilibrium studies. Further correlation between Koiter's theory (using energy theories) and catastrophe theory (using topological similarities) has been suggested by Thompson, (Ref. A11). Hanson, (Ref. A12), has demonstrated that through an extension of Koiter's method, a two-mode buckling problem is shown to be represented by a three-dimensional surface. He classifies this surface as to bifurcation and correlates these with catastrophe theory. This is again illustrated by Hui and Hansen, (Ref. A13), in a recent publication. In a separate publication Hansen and Hui (Ref. A14) show that the potential function for a two mode buckling of a spherical shell can be described as

$$P = x^2 y^2 + y^4 + tx^2 + wy^2 - ux - vy$$

where x and y are amplitudes of certain deflection modes and u, v, w and t are either applied loads or initial imperfections. This is called the parabolic umbilic catastrophe. By applying the theory of Thom (Ref. A15), the question of stability can be resolved. Hansen and Hui show that by examining the stable and unstable forms of the above function, the complete behavior of a sphere subjected to uniform and varying pressures can be mapped.

Alesso [A16] has further demonstrated how the mapping of a control parameter ($\frac{\partial P}{\partial Y}$) onto a three-dimensional surface relates to the possible deformations that can occur in a shallow spherical cap. Figure A1 illustrates the interrelationship between elementary catastrophe theory and that suggested by the more accepted energy methods. Clearly, the two-dimensional plane suggested by energy methods contains less information than that developed from catastrophe theory.

If one were to fully understand the implications of the catastrophe theory, the mathematical description of the potential function developed for structural stability may be more readily understood, particularly for coincident bifurcation loads. For single buckling paths, catastrophe theory does not appear to have an advantage over energy methods. The major advantage that can be utilized by employing catastrophe theory is that a qualitative measure of the effects of imperfections can be obtained. Which imperfection shape and the number of terms required for proper convergence can be identified using this theory. The method is not quantitative. The key here is understanding some new mathematical development. At present, the newly trained design engineer would probably not have the discipline nor interest. However, it may be time for the engineer to open his eyes and possibly develop the "by hook or by crook" method Koiter was alluding to.

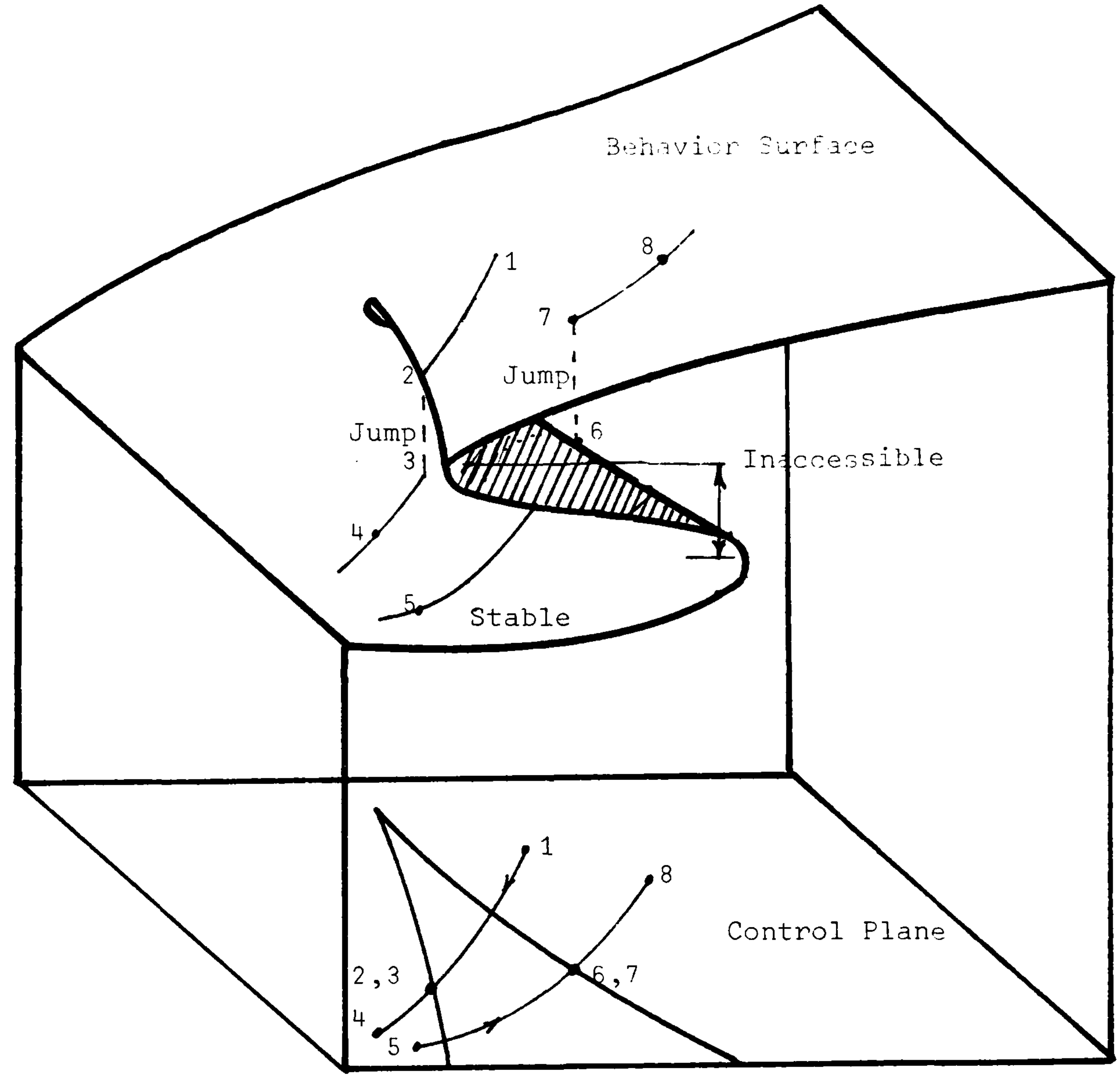

Figure A-1. Comparative surfaces for catastrophe theory and energy methods of a shallow spherical cap.

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